Introduction to Real Analysis

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About This Document

I often teach the MATH 501-502: Introduction to Real Analysis course at the University of Louisville. The course is intended for a mix of mostly upper-level mathematics majors with a smattering of other students from physics and engineering. These are notes I've compiled over the years. They cover the basic ideas of analysis on the real line.

Prerequisites are a good calculus course, including standard differentiation and integration methods for real functions, and a course in which the students must read and write proofs. Some familiarity with basic set theory and standard proof methods such as induction and contradiction is needed. The most important thing is some mathematical sophistication beyond the basic algorithm- and computation-based courses.

Feel free to use these notes for any purpose, as long as you give me blame or credit. In return, I only ask you to tell me about mistakes. Any suggestions for improvements and additions are very much appreciated. I can be contacted using the email address on the Web page referenced below.

The notes are updated and corrected quite often. The date of the version you're reading is at the bottom-left of most pages. The latest version is available for download at the Web address `math.louisville.edu/∼lee/ira`.

There are many exercises at the ends of the chapters. There is no general collection of solutions.

Some early versions of the notes leaked out onto the Internet and they are being offered by a few of the usual download sites. The early versions were meant for me and my classes only, and contain many typos and a few — gasp! — outright mistakes. Please help me expunge those escapees from the Internet by pointing those who offer the older files to the latest version.

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Chapter 1

Basic Ideas

In the end, all mathematics can be boiled down to logic and set theory. Because of this, any careful presentation of fundamental mathematical ideas is inevitably couched in the language of logic and sets. This chapter defines enough of that language to allow the presentation of basic real analysis. Much of it will be familiar to you, but look at it anyway to make sure you understand the notation.

1.1 Sets

Set theory is a large and complicated subject in its own right. There is no time in this course to touch on any but the simplest parts of it. Instead, we'll just look at a few topics from what is often called "naive set theory," many of which should already be familiar to you.

We begin with a few definitions.

A *set* is a collection of objects called *elements.* Usually, sets are denoted by the capital letters $A, B, \cdots, Z$. A set can consist of any type and number of elements. Even other sets can be elements of a set. The sets dealt with here usually have real numbers as their elements.

If a is an element of the set A, we write $a \in A$. If a is not an element of the set A, we write $a \notin A$.

If all the elements of A are also elements of B, then A is a *subset* of B. In this case, we write $A \subset B$ or $B \supset A$. In particular, notice that whenever A is a set, then $A \subset A$.

Two sets A and B are *equal*, if they have the same elements. In this case we write $A = B$. It is easy to see that $A = B$ iff $A \subset B$ and $B \subset A$. Establishing that both of these containments are true is the most common way to show two sets are equal.

If $A \subset B$ and $A \neq B$, then A is a *proper* subset of B. In cases when this is important, it is written $A \subsetneqq B$ instead of just $A \subset B$.

There are several ways to describe a set.

A set can be described in words such as "P is the set of all presidents of the United States." This is cumbersome for complicated sets.

All the elements of the set could be listed in curly braces as $S = \{2, 0, a\}$. If the set has many elements, this is impractical, or impossible.

More common in mathematics is set builder notation. Some examples are

$$\begin{aligned} P &= \{p : p \text{ is a president of the United states}\} \\ &= \{\text{Washington, Adams, Jefferson}, \cdots, \text{Bush, Obama, Trump, Biden}\} \end{aligned}$$

and

$$A = \{n : n \text{ is a prime number}\} = \{2, 3, 5, 7, 11, \cdots\}.$$

In general, the set builder notation defines a set in the form

$$\{\text{formula for a typical element} : \text{objects to plug into the formula}\}.$$

A more complicated example is the set of perfect squares:

$$S = \{n^2 : n \text{ is an integer}\} = \{0, 1, 4, 9, \cdots\}.$$

The existence of several sets will be assumed.

The simplest of these is the *empty set*, which is the set with no elements. It is denoted as $\emptyset$. The *natural numbers* is the set $\mathbb{N} = \{1, 2, 3, \cdots\}$ consisting of the positive integers. The set $\mathbb{Z} = \{\cdots, -2, -1, 0, 1, 2, \cdots\}$ is the set of all integers. $\omega = \{n \in \mathbb{Z} : n \geq 0\} = \{0, 1, 2, \cdots\}$ is the nonnegative integers. Clearly, $\emptyset \subset A$, for any set A and

$$\emptyset \subset \mathbb{N} \subset \omega \subset \mathbb{Z}.$$

Definition 1.1. Given any set A, the *power set* of A, written $\mathcal{P}(A)$, is the set consisting of all subsets of A; i.e.,

$$\mathcal{P}(A) = \{B : B \subset A\}.$$

For example, $\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Also, for any set A, it is always true that $\emptyset \in \mathcal{P}(A)$ and $A \in \mathcal{P}(A)$. If $a \in A$, it is rarely true that

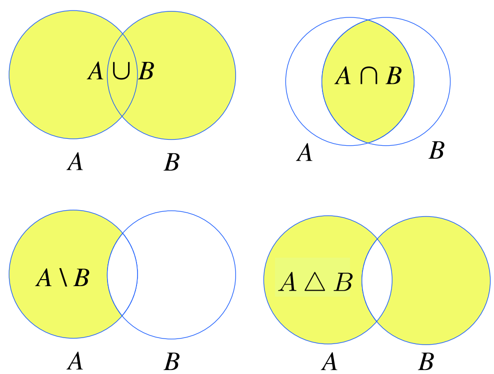


Figure 1.1: These are Venn diagrams showing the four standard binary operations on sets. In this figure, the set which results from the operation is shaded.

$a \in \mathcal{P}(A)$, but it is always true that $\{a\} \subset \mathcal{P}(A)$. Make sure you understand why!

An amusing example is $\mathcal{P}(\emptyset) = \{\emptyset\}$. (Don't confuse $\emptyset$ with $\{\emptyset\}$! The former is empty and the latter has one element.) Now, consider

$$\begin{aligned}
&\mathcal{P}(\emptyset) = \{\emptyset\} \\
&\mathcal{P}(\mathcal{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\} \\
&\mathcal{P}(\mathcal{P}(\mathcal{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}
\end{aligned}$$

After continuing this n times, for some $n \in \mathbb{N}$, the resulting set,

$$\mathcal{P}(\mathcal{P}(\cdots \mathcal{P}(\emptyset) \cdots)),$$

is very large. In fact, since a set with k elements has 2^k elements in its power set, there are $2^{2^{2^2}} = 65,536$ elements after only five iterations of the example. After only a few more iterations, the result gets too large to print. Number constructions such as this one are sometimes called *tetrations*.

1.2 Algebra of Sets

Let A and B be sets. There are four common binary operations used on sets.

The *union* of A and B is the set containing all the elements in either A or B:

$$A \cup B = \{x : x \in A \text{ or } x \in B\}.$$

The *intersection* of A and B is the set containing the elements contained in both A and B:

$$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

The *difference* of A and B is the set of elements in A and not in B:

$$A \setminus B = \{x : x \in A \text{ and } x \notin B\}.$$

The *symmetric difference* of A and B is the set of elements in one of the sets, but not both:

$$A \bigtriangleup B = (A \cup B) \setminus (A \cap B).$$

Another common set operation is *complementation*. The complement of a set A is usually thought of as the set consisting of all elements which are not in A. But, a little thinking will convince you this is not a meaningful definition because the collection of elements not in A is not a precisely understood collection. To make sense of the complement of a set, there must be a well-defined *universal set* U which contains all the sets in question. Then the *complement* of a set $A \subset U$ is $A^c = U \setminus A$. It is usually the case that the universal set U is evident from the context in which it is used.

With these operations, an extensive algebra for the manipulation of sets can be developed. It's usually done hand in hand with formal logic because the two subjects share much in common. These topics are studied as part of *Boolean algebra.*[1] Several examples of set algebra are given in the following theorem and its corollary.

Theorem 1.2. *Let A, B and C be sets.*

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof. (a) This is proved as a sequence of equivalences.[2]

$$\begin{aligned} x \in A \setminus (B \cup C) &\iff x \in A \text{ and } x \notin (B \cup C) \\ &\iff x \in A \text{ and } x \notin B \text{ and } x \notin C \\ &\iff (x \in A \text{ and } x \notin B) \text{ and } (x \in A \text{ and } x \notin C) \\ &\iff x \in (A \setminus B) \cap (A \setminus C) \end{aligned}$$

[1] George Boole (1815-1864)

[2] The logical symbol $\iff$ is the same as "if, and only if." If A and B are any two statements, then $A \iff B$ is the same as saying A implies B and B implies A. It is also common to use *iff* in this way.

(b) This is also proved as a sequence of equivalences.

$$\begin{aligned} x \in A \setminus (B \cap C) &\iff x \in A \text{ and } x \notin (B \cap C) \\ &\iff x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\ &\iff (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\ &\iff x \in (A \setminus B) \cup (A \setminus C) \end{aligned}$$

□

Theorem 1.2 is a version of a pair of set equations which are often called De Morgan's Laws.[3] The more usual statement of De Morgan's Laws is in Corollary 1.3, which is an obvious consequence of Theorem 1.2 when there is a universal set to make complementation well-defined.

Corollary 1.3 (De Morgan's Laws)**.** *Let A and B be sets.*

(a) $(A \cup B)^c = A^c \cap B^c$

(b) $(A \cap B)^c = A^c \cup B^c$

1.3 Indexed Sets

We often have occasion to work with large collections of sets. For example, we could have a sequence of sets $A_1, A_2, A_3, \cdots$, where there is a set A_n associated with each $n \in \mathbb{N}$. In general, let Λ be a set and suppose for each $\lambda \in \Lambda$ there is a set A_λ. The set $\{A_\lambda : \lambda \in \Lambda\}$ is called a *collection of sets indexed by* Λ. In this case, Λ is called the *indexing set* for the collection.

Example 1.1. For each $n \in \mathbb{N}$, let $A_n = \{k \in \mathbb{Z} : k^2 \leq n\}$. Then

$$\begin{aligned} &A_1 = A_2 = A_3 = \{-1, 0, 1\},\ A_4 = \{-2, -1, 0, 1, 2\},\ \cdots, \\ &\qquad A_{61} = \{-7, -6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6, 7\}, \cdots \end{aligned}$$

is a collection of sets indexed by $\mathbb{N}$.

Two of the basic binary operations can be extended to work with indexed collections. In particular, using the indexed collection from the previous paragraph, we define

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for some } \lambda \in \Lambda\}$$

[3] Augustus De Morgan (1806–1871)

and

$$\bigcap_{\lambda\in\Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for all } \lambda \in \Lambda\}.$$

De Morgan's Laws can be generalized to indexed collections.

Theorem 1.4. *If $\{B_\lambda : \lambda \in \Lambda\}$ is an indexed collection of sets and A is a set, then*

$$A \setminus \bigcup_{\lambda\in\Lambda} B_\lambda = \bigcap_{\lambda\in\Lambda} (A \setminus B_\lambda)$$

and

$$A \setminus \bigcap_{\lambda\in\Lambda} B_\lambda = \bigcup_{\lambda\in\Lambda} (A \setminus B_\lambda).$$

Proof. The proof of this theorem is Exercise 4. □

1.4 Functions and Relations

1.4.1 Tuples

When listing the elements of a set, the order in which they are listed is unimportant; e.g., $\{e, l, v, i, s\} = \{l, i, v, e, s\}$. If the order in which n items are listed is important, the list is called an *n-tuple*. (Strictly speaking, an n-tuple is not a set.) We denote an n-tuple by enclosing the ordered list in parentheses. For example, if x_1, x_2, x_3, x_4 are four items, the 4-tuple (x_1, x_2, x_3, x_4) is different from the 4-tuple (x_2, x_1, x_3, x_4).

Because they are used so often, the cases when $n = 2$ and $n = 3$ have special names: 2-tuples are called *ordered pairs* and a 3-tuple is called an *ordered triple*.

Definition 1.5. Let A and B be sets. The set of all ordered pairs

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$$

is called the *Cartesian product* of A and B.[4]

Example 1.2. If $A = \{a, b, c\}$ and $B = \{1, 2\}$, then

$$A \times B = \{(a, 1), (a, 2), (b, 1), (b, 2), (c, 1), (c, 2)\}.$$

and

$$B \times A = \{(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)\}.$$

Notice that $A \times B \neq B \times A$ because of the importance of *order* in the ordered pairs.

[4] René Descartes (1596–1650)

A useful way to visualize the Cartesian product of two sets is as a table. The Cartesian product $A \times B$ from Example 1.2 is listed as the entries of the following table.

$\times$	1	2
a	$(a,1)$	$(a,2)$
b	$(b,1)$	$(b,2)$
c	$(c,1)$	$(c,2)$

Of course, the common Cartesian plane from your analytic geometry course is nothing more than a way of visualizing the Cartesian product of the real numbers with real numbers.

The definition of Cartesian product can be extended to the case of more than two sets. If $\{A_1, A_2, \cdots, A_n\}$ are sets, then

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_k \in A_k \text{ for } 1 \leq k \leq n\}$$

is a set of n-tuples. This is often written as

$$\prod_{k=1}^{n} A_k = A_1 \times A_2 \times \cdots \times A_n.$$

1.4.2 Relations

Definition 1.6. If A and B are sets, then any $R \subset A \times B$ is a *relation from A to B*. If $(a,b) \in R$, we write aRb.

In this case,

$$\operatorname{dom}(R) = \{a : (a,b) \in R \text{ for some } b\} \subset A$$

is the *domain* of R and

$$\operatorname{ran}(R) = \{b : (a,b) \in R \text{ for some } a\} \subset B$$

is the *range* of R. It may happen that $\operatorname{dom}(R)$ and $\operatorname{ran}(R)$ are proper subsets of A and B, respectively.

In the special case when $R \subset A \times A$, for some set A, there is some additional terminology.

R is *symmetric*, if $aRb \iff bRa$.

R is *reflexive*, if aRa whenever $a \in \operatorname{dom}(R)$.

R is *transitive*, if aRb and $bRc \implies aRc$.

R is an *equivalence relation* on A, if it is symmetric, reflexive and transitive.

Example 1.3. Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $aRb \iff a \le b$. Then R is reflexive and transitive, but not symmetric.

Example 1.4. Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $aRb \iff a < b$. Then R is transitive, but neither reflexive nor symmetric.

Example 1.5. Let R be the relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $aRb \iff a^2 = b^2$. In this case, R is an equivalence relation. It is evident that aRb iff $b = a$ or $b = -a$.

1.4.3 Functions

Definition 1.7. A relation $R \subset A \times B$ is a *function* if

$$aRb_1 \text{ and } aRb_2 \iff b_1 = b_2.$$

If $f \subset A \times B$ is a function and $\operatorname{dom}(f) = A$, then we usually write $f : A \to B$ and use the usual notation $f(a) = b$ instead of afb.

If $f : A \to B$ is a function, the usual intuitive interpretation is to regard f as a rule that associates each element of A with a unique element of B. It's not necessarily the case that each element of B is associated with something from A; i.e., B may not be $\operatorname{ran}(f)$. It's also common for more than one element of A to be associated with the same element of B.

Example 1.6. Define $f : \mathbb{N} \to \mathbb{Z}$ by $f(n) = n^2$ and $g : \mathbb{Z} \to \mathbb{Z}$ by $g(n) = n^2$. In this case $\operatorname{ran}(f) = \{n^2 : n \in \mathbb{N}\}$ and $\operatorname{ran}(g) = \operatorname{ran}(f) \cup \{0\}$. Notice that even though f and g use the same formula, they are actually different functions.

Definition 1.8. If $f : A \to B$ and $g : B \to C$, then the *composition* of g with f is the function $g \circ f : A \to C$ defined by $g \circ f(a) = g(f(a))$.

In Example 1.6, $g \circ f(n) = g(f(n)) = g(n^2) = (n^2)^2 = n^4$ makes sense for all $n \in \mathbb{N}$, but $f \circ g$ is undefined at $n = 0$.

There are several important types of functions.

Definition 1.9. A function $f : A \to B$ is a *constant function*, if $\operatorname{ran}(f)$ has a single element; i.e., there is a $b \in B$ such that $f(a) = b$ for all $a \in A$. The function f is *surjective* (or *onto* B), if $\operatorname{ran}(f) = B$.

In a sense, constant and surjective functions are the opposite extremes. A constant function has the smallest possible range and a surjective function has the largest possible range. Of course, a function $f : A \to B$ can be both constant and surjective, if B has only one element.

Definition 1.10. A function $f : A \to B$ is *injective* (or *one-to-one*), if $f(a) = f(b)$ implies $a = b$.

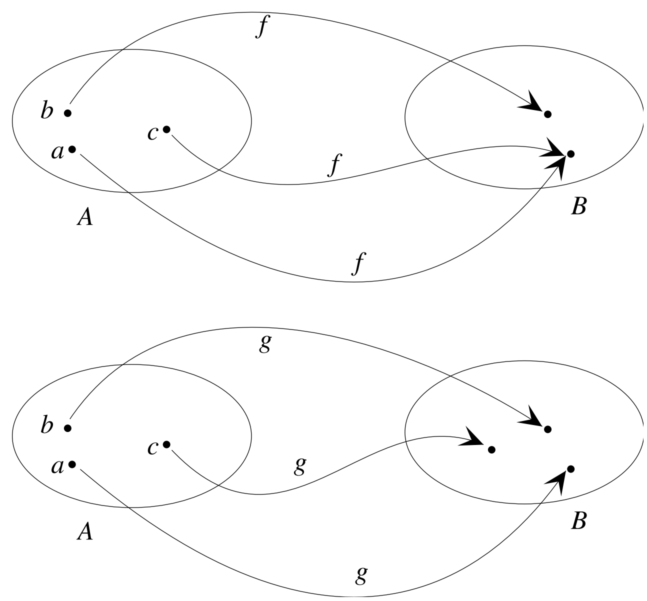


Figure 1.2: These diagrams show two functions, $f : A \to B$ and $g : A \to B$. The function g is injective and f is not because $f(a) = f(c)$.

The terminology "one-to-one" is very descriptive because such a function uniquely pairs up the elements of its domain and range. An illustration of this definition is in Figure 1.2. In Example 1.6, f is injective while g is not.

Definition 1.11. A function $f : A \to B$ is *bijective*, if it is both surjective and injective.

A bijective function can be visualized as uniquely pairing up all the elements of A and B. Some authors, favoring less pretentious language, use the more descriptive terminology *one-to-one correspondence* instead of bijection. This pairing up of the elements from each set is like counting them and finding they have the same number of elements. Given any two sets, no matter how many elements they have, the intuitive idea is they have the same number of elements if, and only if, there is a bijection between them.

The following theorem shows that this property of counting the number of elements works in a familiar way. (Its proof is left as Exercise 8.)

Theorem 1.12. *If $f : A \to B$ and $g : B \to C$ are bijections, then $g \circ f : A \to C$ is a bijection.*

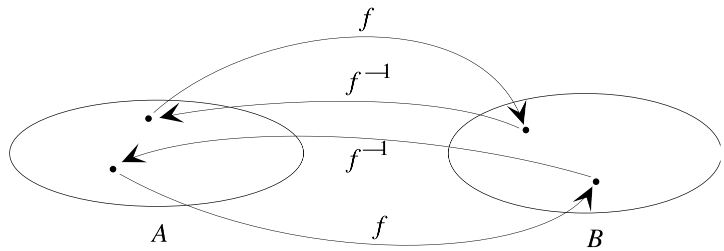


Figure 1.3: This is one way to visualize a general invertible function. First f does something to a and then f^{-1} undoes it.

1.4.4 Inverse Functions

Definition 1.13. If $f : A \to B$, $C \subset A$ and $D \subset B$, then the *image* of C is the set $f(C) = \{f(a) : a \in C\}$. The *inverse image* of D is the set $f^{-1}(D) = \{a : f(a) \in D\}$.

Definitions 1.11 and 1.13 work together in the following way. Suppose $f : A \to B$ is bijective and $b \in B$. The fact that f is surjective guarantees that $f^{-1}(\{b\}) \neq \emptyset$. Since f is injective, $f^{-1}(\{b\})$ contains only one element, say a, where $f(a) = b$. In this way, it is seen that f^{-1} is a rule that assigns each element of B to exactly one element of A; i.e., f^{-1} is a function with domain B and range A.

Definition 1.14. If $f : A \to B$ is bijective, the *inverse* of f is the function $f^{-1} : B \to A$ with the property that $f^{-1} \circ f(a) = a$ for all $a \in A$ and $f \circ f^{-1}(b) = b$ for all $b \in B$.[5]

There is some ambiguity in the meaning of f^{-1} between 1.13 and 1.14. The former is an operation working with subsets of A and B; the latter is a function working with elements of A and B. It's usually clear from the context which meaning is being used.

Example 1.7. Let $A = \mathbb{N}$ and B be the even natural numbers. If $f : A \to B$ is $f(n) = 2n$ and $g : B \to A$ is $g(n) = n/2$, it is clear f is bijective. Since $f \circ g(n) = f(n/2) = 2n/2 = n$ and $g \circ f(n) = g(2n) = 2n/2 = n$, we see $g = f^{-1}$. (Of course, it is also true that $f = g^{-1}$.)

[5]The notation $f^{-1}(x)$ for the inverse is unfortunate because it is so easily confused with the multiplicative inverse, $(f(x))^{-1}$. For a discussion of this, see [9]. The context is usually enough to avoid confusion.

Example 1.8. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined by

$$f(n) = \begin{cases} (n-1)/2, & n \text{ odd}, \\ -n/2, & n \text{ even} \end{cases}$$

It's quite easy to see that f is bijective and

$$f^{-1}(n) = \begin{cases} 2n+1, & n \geq 0, \\ -2n, & n < 0 \end{cases}$$

Given any set A, it's obvious there is a bijection $f : A \to A$ and, if $g : A \to B$ is a bijection, then so is $g^{-1} : B \to A$. Combining these observations with Theorem 1.12, an easy theorem follows.

Theorem 1.15. *Let $\mathcal{S}$ be a collection of sets. The relation on $\mathcal{S}$ defined by*

$$A \sim B \iff \text{there is a bijection } f : A \to B$$

is an equivalence relation.

1.4.5 Schröder-Bernstein Theorem

Suppose A and B are disjoint sets such that the number of elements in A is no more than the number of elements in B and the number of elements in B is no more than the number of elements in A. Intuition tells us that both sets must have the same number of elements. Our intuition is based on the finite sets we see every day, and if the sets are finite, then a simple proof using the pigeonhole principle shows the two sets indeed have the same number of elements. Unfortunately, intuition sometimes fails when sets aren't finite. The following theorem confirms our intuition for sets of any size. Its proof is quite tricky.

Theorem 1.16 (Schröder-Bernstein[6])**.** *Let A and B be sets. If there are injective functions $f : A \to B$ and $g : B \to A$, then there is a bijective function $h : A \to B$.*

Proof. Let $B_1 = B \setminus f(A)$. If $B_k \subset B$ is defined for some $k \in \mathbb{N}$, let $A_k = g(B_k)$ and $B_{k+1} = f(A_k)$. This inductively defines A_k and B_k for all $k \in \mathbb{N}$. Use these

[6]Felix Bernstein (1878–1956), Ernst Schröder (1841–1902).
This is often called the Cantor-Schröder-Bernstein or Cantor-Bernstein Theorem after Georg Cantor (1845–1918), despite the fact that it was apparently first proved by Richard Dedekind (1831–1916).

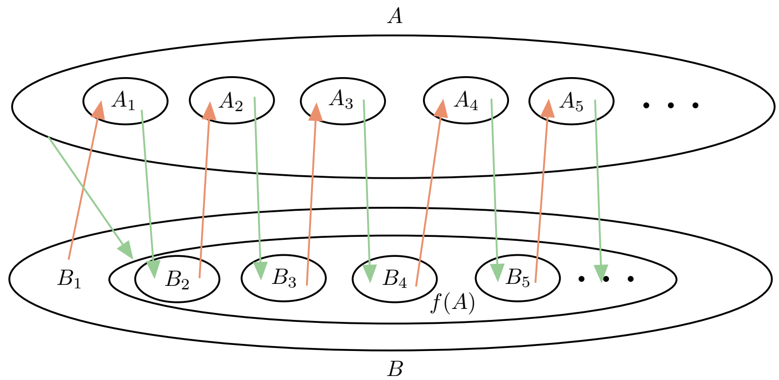


Figure 1.4: Here are the first few steps from the construction used in the proof of Theorem 1.16.

sets to define $\tilde{A} = \bigcup_{k\in\mathbb{N}} A_k$ and $h : A \to B$ as

$$h(x) = \begin{cases} g^{-1}(x), & x \in \tilde{A} \\ f(x), & x \in A \setminus \tilde{A} \end{cases}.$$

It must be shown that h is well-defined, injective and surjective.

To show h is well-defined, let $x \in A$. If $x \in A \setminus \tilde{A}$, then it is clear $h(x) = f(x)$ is defined. On the other hand, if $x \in \tilde{A}$, then $x \in A_k$ for some k. Since $x \in A_k = g(B_k)$, we see $h(x) = g^{-1}(x)$ is defined. Therefore, h is well-defined.

To show h is injective, let $x, y \in A$ with $x \neq y$.

If both $x, y \in \tilde{A}$ or $x, y \in A \setminus \tilde{A}$, then the assumptions that g and f are injective, respectively, imply $h(x) \neq h(y)$.

The remaining case is when $x \in \tilde{A}$ and $y \in A \setminus \tilde{A}$. Suppose $x \in A_k$ and $h(x) = h(y)$. If $k = 1$, then $h(x) = g^{-1}(x) \in B_1$ and $h(y) = f(y) \in f(A) = B \setminus B_1$. This is clearly incompatible with the assumption that $h(x) = h(y)$. Now, suppose $k > 1$. Then there is an $x_1 \in B_1$ such that

$$x = \underbrace{g \circ f \circ g \circ f \circ \cdots \circ f \circ g}_{k-1\ f\text{'s and } k\ g\text{'s}}(x_1).$$

This implies

$$h(x) = g^{-1}(x) = \underbrace{f \circ g \circ f \circ \cdots \circ f \circ g}_{k-1\ f\text{'s and } k-1\ g\text{'s}}(x_1) = f(y)$$

so that

$$y = \underbrace{g \circ f \circ g \circ f \circ \cdots \circ f \circ g}_{k-2\ f\text{'s and } k-1\ g\text{'s}}(x_1) \in A_{k-1} \subset \tilde{A}.$$

This contradiction shows that $h(x) \neq h(y)$. We conclude h is injective.

To show h is surjective, let $y \in B$. If $y \in B_k$ for some k, then $h(A_k) = g^{-1}(A_k) = B_k$ shows $y \in h(A)$. If $y \notin B_k$ for any k, $y \in f(A)$ because $B_1 = B \setminus f(A)$, and $g(y) \notin \tilde{A}$, so $y = h(x) = f(x)$ for some $x \in A$. This shows h is surjective. □

The Schröder-Bernstein theorem has many consequences, some of which are at first a bit unintuitive, such as the following theorem.

Corollary 1.17. *There is a bijective function $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$*

Proof. If $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is $f(n) = (n, 1)$, then f is clearly injective. On the other hand, suppose $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined by $g((a, b)) = 2^a 3^b$. The uniqueness of prime factorizations guarantees g is injective. An application of Theorem 1.16 yields h. □

To appreciate the power of the Schröder-Bernstein theorem, try to find an explicit bijection $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

1.5 Cardinality

There is a way to use sets and functions to formalize and generalize how we count. For example, suppose we want to count how many elements are in the set $\{a, b, c\}$. The natural way to do this is to point at each element in succession and say "one, two, three." What we're doing is defining a bijective function between $\{a, b, c\}$ and the set $\{1, 2, 3\}$. This idea can be generalized.

Definition 1.18. Given $n \in \omega$, the set $\overline{n} = \{0, 1, 2, \cdots, n-1\}$ is called an *initial segment* of ω. A set S has *cardinality* n, if there is a bijective function $f : S \to \overline{n}$. In this case, we write $\operatorname{card}(S) = n$.

The cardinalities defined in Definition 1.18 are called the *finite* cardinal numbers. They correspond to the everyday counting numbers we usually use. The idea can be generalized still further.

Definition 1.19. Let A and B be two sets. If there is an injective function $f : A \to B$, we say $\operatorname{card}(A) \leq \operatorname{card}(B)$.

According to Theorem 1.16, the Schröder-Bernstein Theorem, if $\operatorname{card}(A) \leq \operatorname{card}(B)$ and $\operatorname{card}(B) \leq \operatorname{card}(A)$, then there is a bijective function $f : A \to B$. As expected, in this case we write $\operatorname{card}(A) = \operatorname{card}(B)$. When $\operatorname{card}(A) \leq \operatorname{card}(B)$, but no such bijection exists, we write $\operatorname{card}(A) < \operatorname{card}(B)$. Theorem 1.15 shows that $\operatorname{card}(A) = \operatorname{card}(B)$ is an equivalence relation between sets.

The idea here, of course, is that $\operatorname{card}(A) = \operatorname{card}(B)$ means A and B have the same number of elements and $\operatorname{card}(A) < \operatorname{card}(B)$ means A is a smaller set than B. We'll see this simple intuitive understanding has some surprising consequences when the sets are infinite.

In particular, the set A is *countable*, if $\operatorname{card}(A) \leq \operatorname{card}(\mathbb{N})$. A countable set A is said to be *countably infinite* when $\operatorname{card}(A) = \operatorname{card}(\mathbb{N})$ and *finite* when $\operatorname{card}(A) < \operatorname{card}(\mathbb{N})$. In other words, the countable sets are those having finite or countable cardinality. It is also common to abbreviate $\operatorname{card}(\mathbb{N}) = \aleph_0$.[7]

Example 1.9. Let $f : \mathbb{N} \to \mathbb{Z}$ be defined as

$$f(n) = \begin{cases} \frac{n+1}{2}, & \text{when } n \text{ is odd} \\ 1 - \frac{n}{2}, & \text{when } n \text{ is even} \end{cases}.$$

It's easy to show f is a bijection, so $\operatorname{card}(\mathbb{N}) = \operatorname{card}(\mathbb{Z}) = \aleph_0$.

Theorem 1.20. (a) *If A and B are countable sets, then so is $A \times B$.*
(b) *If $\{A_n : n \in \mathbb{N}\}$ is a collection of countable sets, then $\bigcup_{n\in\mathbb{N}} A_n$ is countable.*

Proof. (a) This is a consequence of Corollary 1.17.
(b) This is Exercise 24. □

An alert reader will have noticed from previous examples that

$$\aleph_0 = \operatorname{card}(\mathbb{Z}) = \operatorname{card}(\omega) = \operatorname{card}(\mathbb{N}) = \operatorname{card}(\mathbb{N}\times\mathbb{N}) = \operatorname{card}(\mathbb{N}\times\mathbb{N}\times\mathbb{N}) = \cdots$$

A logical question is whether all sets either have finite cardinality, or are countably infinite. That this is not so is seen by letting $S = \mathbb{N}$ in the following theorem.

Theorem 1.21 (Cantor[8]). *If S is a set,* $\operatorname{card}(S) < \operatorname{card}(\mathcal{P}(S))$.

Proof. Noting that

$$0 = \operatorname{card}(\emptyset) < 1 = \operatorname{card}(\{\emptyset\}) = \operatorname{card}(\mathcal{P}(\emptyset)),$$

the theorem is apparently true when S is empty.

Suppose $S \neq \emptyset$. Since $\{a\} \in \mathcal{P}(S)$ for all $a \in S$, it follows that $\operatorname{card}(S) \leq \operatorname{card}(\mathcal{P}(S))$. Therefore, it suffices to prove there is no surjective function $f : S \to \mathcal{P}(S)$.

[7]The symbol $\aleph$ is *aleph*, the first letter of the Hebrew alphabet. $\aleph_0$ is usually pronounced "aleph naught." It is called the first infinite cardinal number.

[8]Georg Cantor (1845–1918)

To see this, assume there is such a function f and let $T = \{x \in S : x \notin f(x)\}$. Since f is surjective, there is a $t \in S$ such that $f(t) = T$. Either $t \in T$ or $t \notin T$.

If $t \in T = f(t)$, then the definition of T implies $t \notin T$, a contradiction. On the other hand, if $t \notin T = f(t)$, then the definition of T implies $t \in T$, another contradiction. These contradictions lead to the conclusion that no such function f can exist. $\square$

A set S is said to be *uncountably infinite*, or just *uncountable*, if $\aleph_0 < \operatorname{card}(S)$. Theorem 1.21 implies $\aleph_0 < \operatorname{card}(\mathcal{P}(\mathbb{N}))$, so $\mathcal{P}(\mathbb{N})$ is uncountable. In fact, the same argument implies

$$\aleph_0 = \operatorname{card}(\mathbb{N}) < \operatorname{card}(\mathcal{P}(\mathbb{N})) < \operatorname{card}(\mathcal{P}(\mathcal{P}(\mathbb{N}))) < \cdots$$

So, there are an infinite number of distinct infinite cardinalities.

In 1874 Georg Cantor [7] proved $\operatorname{card}(\mathbb{R}) = \operatorname{card}(\mathcal{P}(\mathbb{N})) > \aleph_0$, where $\mathbb{R}$ is the set of real numbers. (A version of Cantor's theorem is Theorem 2.28 below.) This naturally led to the question whether there are sets S such that $\aleph_0 < \operatorname{card}(S) < \operatorname{card}(\mathbb{R})$. Cantor spent many years trying to answer this question and never succeeded. His assumption that no such sets exist came to be called the *continuum hypothesis.*

The importance of the continuum hypothesis was highlighted by David Hilbert[9] at the 1900 International Congress of Mathematicians in Paris, when he put it first on his famous list of the 23 most important open problems in mathematics. Kurt Gödel[10] proved in 1940 that the continuum hypothesis cannot be disproved using standard set theory, but he did not prove it was true. In 1963 it was proved by Paul Cohen[11] that the continuum hypothesis is actually unprovable as a theorem in standard set theory.

So, the continuum hypothesis is a statement with the strange property that it is neither true nor false within the framework of ordinary set theory. This means that in the standard axiomatic development of set theory, the continuum hypothesis, or a careful negation of it, can be taken as an additional axiom without causing any contradictions. The technical terminology is that the continuum hypothesis is *independent* of the axioms of set theory.

The proofs of these theorems are extremely difficult and entire broad areas of mathematics were invented just to make their proofs possible. Even today, there are some deep philosophical questions swirling around them. A more technical introduction to many of these ideas is contained in the book by

[9] David Hilbert (1862–1943)

[10] Kurt Gödel (1906–1978)

[11] Paul Cohen (1934–2007)

Ciesielski [10]. A nontechnical and very readable history of the efforts by mathematicians to understand the continuum hypothesis is the book by Aczel [1]. A shorter, nontechnical account of Cantor's work is in an article by Dauben [11].

1.6 Exercises

Exercise 1.1. If a set S has n elements for $n \in \omega$, then how many elements are in $\mathcal{P}(S)$?

Exercise 1.2. Is there a set S such that $S \cap \mathcal{P}(S) \neq \emptyset$?

Exercise 1.3. Prove that for any sets A and B,

(a) $A = (A \cap B) \cup (A \setminus B)$

(b) $A \cup B = (A \setminus B) \cup (B \setminus A) \cup (A \cap B)$ and that the sets $A \setminus B$, $B \setminus A$ and $A \cap B$ are pairwise disjoint.

(c) $A \setminus B = A \cap B^c$.

Exercise 1.4. Prove Theorem 1.4.

Exercise 1.5. For any sets A, B, C and D,

$$(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$$

and

$$(A \times B) \cup (C \times D) \subset (A \cup C) \times (B \cup D).$$

Why does equality not hold in the second expression?

Exercise 1.6. Prove Theorem 1.15.

Exercise 1.7. Suppose R is an equivalence relation on A. For each $x \in A$ define $C_x = \{y \in A : xRy\}$. Prove that if $x, y \in A$, then either $C_x = C_y$ or $C_x \cap C_y = \emptyset$. (The collection $\{C_x : x \in A\}$ is the set of *equivalence classes* induced by R.)

Exercise 1.8. If $f : A \to B$ and $g : B \to C$ are bijections, then so is $g \circ f : A \to C$.

Exercise 1.9. Prove or give a counter example: $f : X \to Y$ is injective iff whenever A and B are disjoint subsets of Y, then $f^{-1}(A) \cap f^{-1}(B) = \emptyset$.

Exercise 1.10. If $f : A \to B$ is bijective, then f^{-1} is unique.

Exercise 1.11. Prove that $f : X \to Y$ is surjective iff for each subset $A \subset X$, $Y \setminus f(A) \subset f(X \setminus A)$.

Exercise 1.12. Suppose that A_k is a set for each positive integer k.

(a) Show that $x \in \bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$ iff $x \in A_k$ for infinitely many sets A_k.

(b) Show that $x \in \bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right)$ iff $x \in A_k$ for all but finitely many of the sets A_k.

The set $\bigcap_{n=1}^{\infty} \left(\bigcup_{k=n}^{\infty} A_k \right)$ from (a) is often called the *superior limit* of the sets A_k and $\bigcup_{n=1}^{\infty} \left(\bigcap_{k=n}^{\infty} A_k \right)$ is often called the *inferior limit* of the sets A_k.

Exercise 1.13. Given two sets A and B, it is common to let A^B denote the set of all functions $f : B \to A$. Prove that for any set A, $\operatorname{card}\left(\overline{2}^A\right) = \operatorname{card}\left(\mathcal{P}(A)\right)$. This is why many authors use 2^A as their notation for $\mathcal{P}(A)$.

Exercise 1.14. Let S be a set. Prove the following two statements are equivalent:

(a) S is infinite; and,

(b) there is a proper subset T of S and a bijection $f : S \to T$.

This statement is often used as the definition of when a set is infinite.

Exercise 1.15. If S is an infinite set, then there is a countably infinite collection of nonempty pairwise disjoint infinite sets T_n, $n \in \mathbb{N}$ such that $S = \bigcup_{n \in \mathbb{N}} T_n$.

Exercise 1.16. Find an explicit bijection $f : [0, 1] \to (0, 1)$.

Exercise 1.17. If $f : [0, \infty) \to (0, \infty)$ and $g : (0, \infty) \to [0, \infty)$ are given by $f(x) = x + 1$ and $g(x) = x$, then the proof of the Schrǒder-Bernstein theorem yields what bijection $h : [0, \infty) \to (0, \infty)$?

Exercise 1.18. Using the notation from the proof of the Schröder-Bernstein Theorem, let $A = [0, \infty)$, $B = (0, \infty)$, $f(x) = x + 1$ and $g(x) = x$. Determine $h(x)$.

Exercise 1.19. Using the notation from the proof of the Schröder-Bernstein Theorem, let $A = \mathbb{N}$, $B = \mathbb{Z}$, $f(n) = n$ and

$$g(n) = \begin{cases} 1 - 3n, & n \le 0 \\ 3n - 1, & n > 0 \end{cases}.$$

Calculate $h(6)$ and $h(7)$.

Exercise 1.20. Suppose that in the statement of the Schröder-Bernstein theorem $A = B = \mathbb{Z}$ and $f(n) = g(n) = 2n$. Following the procedure in the proof yields what function h?

Exercise 1.21. Find a function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{0\}$ such that $f^{-1} = 1/f$.

Exercise 1.22. Find a bijection $f : [0, \infty) \to (0, \infty)$.

Exercise 1.23. If $f : A \to B$ and $g : B \to A$ are functions such that $f \circ g(x) = x$ for all $x \in B$ and $g \circ f(x) = x$ for all $x \in A$, then $f^{-1} = g$.

Exercise 1.24. If $\{A_n : n \in \mathbb{N}\}$ is a collection of countable sets, then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

Exercise 1.25. If $\{A_n : n \in \mathbb{N}\}$ is a collection of countable sets, then $\bigcup_{n \in \mathbb{N}} A_n$ is countable.

Exercise 1.26. If $\aleph_0 \le \operatorname{card}(S))$, then there is an injective function $f : S \to S$ that is not surjective.

Exercise 1.27. If $\operatorname{card}(S) = \aleph_0$, then there is a sequence of pairwise disjoint sets T_n, $n \in \mathbb{N}$ such that $\operatorname{card}(T_n) = \aleph_0$ for every $n \in \mathbb{N}$ and $\bigcup_{n \in \mathbb{N}} T_n = S$.

Chapter 2

The Real Numbers

This chapter concerns what can be thought of as *the rules of the game:* the axioms of the real numbers. These axioms imply all the properties of the real numbers and, in a sense, any set satisfying them is uniquely determined to be the real numbers.

The axioms are presented here as rules without very much justification. Other approaches can be used. For example, a common approach is to begin with the Peano axioms — the axioms of the natural numbers — and build up to the real numbers through several "completions" of the natural numbers. It's also possible to begin with the axioms of set theory to build up the Peano axioms as theorems and then use those to prove our axioms as further theorems. No matter how it's done, there are always some axioms at the base of the structure and the rules for the real numbers are the same, whether they're axioms or theorems.

We choose to start at the top because the other approaches quickly turn into a long and tedious labyrinth of technical exercises without much connection to analysis.

2.1 The Field Axioms

These first six axioms give the arithmetic properties of the real numbers. They are called the *field axioms* because any object satisfying them is called a *field*.

A *field* is a nonempty set $\mathbb{F}$ along with two binary operations,[1] called

[1] Given a set A, a function $f : A \times A \to A$ is called a *binary operation*. In other words, a binary operation is just a function with two arguments. The standard notations of $+(a, b) = a + b$ and $\times(a, b) = a \times b$ are used here. The symbol $\times$ is unfortunately used for both the Cartesian product and the field operation, but the context in which it's used removes the ambiguity.

multiplication $\times : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ and addition $+ : \mathbb{F} \times \mathbb{F} \to \mathbb{F}$ satisfying the following axioms.

Axiom 1 (Associative Laws)**.** If $a, b, c \in \mathbb{F}$, then $(a + b) + c = a + (b + c)$ and $(a \times b) \times c = a \times (b \times c)$.

Axiom 2 (Commutative Laws)**.** If $a, b \in \mathbb{F}$, then $a + b = b + a$ and $a \times b = b \times a$.

Axiom 3 (Distributive Law)**.** If $a, b, c \in \mathbb{F}$, then $a \times (b + c) = (a \times b) + (a \times c)$.

Axiom 4 (Existence of identities)**.** There are $0, 1 \in \mathbb{F}$ with $0 \neq 1$ such that $a + 0 = a$ and $a \times 1 = a$, for all $a \in \mathbb{F}$.

Axiom 5 (Existence of an additive inverse)**.** For each $a \in \mathbb{F}$ there is $-a \in \mathbb{F}$ such that $a + (-a) = 0$.

Axiom 6 (Existence of a multiplicative inverse)**.** For each $a \in \mathbb{F} \setminus \{0\}$ there is $a^{-1} \in \mathbb{F}$ such that $a \times a^{-1} = 1$.

Although these axioms seem to contain most properties of the real numbers we normally use, they don't characterize the real numbers; they just give the rules for arithmetic. There are many other fields besides the real numbers and studying them is a large part of many abstract algebra courses.

Example 2.1. From elementary algebra we know that the rational numbers

$$\mathbb{Q} = \{p/q : p \in \mathbb{Z} \text{ and } q \in \mathbb{N}\}$$

form a field. It will be shown in Theorem 2.15 that $\sqrt{2} \notin \mathbb{Q}$, so $\mathbb{Q}$ doesn't contain all the real numbers.

Example 2.2. Let $\mathbb{F} = \{0, 1, 2\}$ with addition and multiplication calculated modulo 3. The addition and multiplication tables are as follows.

$+$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\times$	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

It is easy to check that the field axioms are satisfied. This field is usually called $\mathbb{Z}_3$.

The following theorems, containing just a few useful properties of fields, are presented mostly as examples showing how the axioms are used. More complete developments can be found in any beginning abstract algebra text.

Theorem 2.1. *Let $\mathbb{F}$ be a field.*

(a) *The additive and multiplicative identities of $\mathbb{F}$ are unique.*

(b) *If $a, b \in \mathbb{F}$ with $b \neq 0$, then $-a$ and b^{-1} are unique.*

Proof. Suppose e_1 and e_2 are both multiplicative identities in $\mathbb{F}$. Then, according to Axiom 4,

$$e_1 = e_1 \times e_2 = e_2,$$

so the multiplicative identity is unique. The proof for the additive identity is essentially the same.

Suppose b_1 and b_2 are both multiplicative inverses for $b \neq 0$. Then, using Axioms 4 and 1,

$$b_1 = b_1 \times 1 = b_1 \times (b \times b_2) = (b_1 \times b) \times b_2 = 1 \times b_2 = b_2.$$

This shows the multiplicative inverse in unique. The proof is essentially the same for the additive inverse. $\square$

Theorem 2.2 (Cancellation Laws)**.** *Let $a, b, c \in \mathbb{F}$.*

(a) $a = b \iff a + c = b + c$

(b) *Assume $c \neq 0$. Then $a = b \iff a \times c = b \times c$.*

Proof. To prove (a), first note that if $a = b$, then obviously $a + c = b + c$. To prove the other direction, note that

$$\begin{aligned}
a + c = b + c &\implies a + c + (-c) = b + c + (-c) && \text{From above.}\\
&\implies a + (c + (-c)) = b + (c + (-c)) && \text{Axiom 1}\\
&\implies a + 0 = b + 0 && \text{Axiom 5}\\
&\implies a = b && \text{Axiom 4}
\end{aligned}$$

and case (a) follows.

The multiplication case is proved similarly. $\square$

Theorem 2.3. *Let $\mathbb{F}$ be a field with $a, b \in \mathbb{F}$.*

(a) $a \times 0 = 0$

(b) $-(-a) = a$

(c) *If $a \neq 0$, $(a^{-1})^{-1} = a$*

(d) $(-a) \times b = a \times (-b) = -(a \times b)$

(e) $(-a) \times (-b) = -(a \times (-b)) = a \times b$

Proof. To prove (a), note that

$$a \times 0 = a \times (0 + 0) = a \times 0 + a \times 0,$$

where Axiom 1 and Axiom 4 have been used. Now use Theorem 2.2 to see $a \times 0 = 0$.

To prove (b), note

$$\begin{aligned} -(-a) &= -(-a) + 0 = -(-a) + (-a + a) \\ &= (-(-a) + (-a)) + a = 0 + a = a, \end{aligned}$$

where Axioms 1 and 4 were used.

Part (c) is proved similarly to (b).

Part (d) is Exercise 1 at the end of this chapter.

Part (e) follows from (d) and (b) because

$$(-a) \times (-b) = -(a \times (-b)) = -(-(a \times b)) = a \times b.$$

□

There are many other properties of fields which could be proved here, but they correspond to the usual properties of the arithmetic learned in elementary school, and more properly belong to an abstract algebra course, so we omit them. Some of them are in the exercises.

From now on, the standard notations for algebra will usually be used; e. g., we will allow ab instead of $a \times b$, $a - b$ for $a + (-b)$ and a/b instead of $a \times b^{-1}$. We assume as true the standard facts about arithmetic learned in elementary algebra courses.

2.2 The Order Axiom

The axiom of this section gives the order and metric properties of the real numbers. In a sense, the following axiom adds some geometry to a field.

Axiom 7 (Order axiom.)**.** There is a set $P \subset \mathbb{F}$ such that

(a) If $a, b \in P$, then $a + b, ab \in P$.[2]

[2]Algebra texts would say is P is *closed* under addition and multiplication. In Chapter 5 we'll use the word "closed" with a different meaning. This is one of the cases where algebraists and analysts speak different languages. Fortunately, the context usually erases confusion.

(b) If $a \in \mathbb{F}$, then exactly one of the following is true:

$$a \in P, \ -a \in P \text{ or } a = 0.$$

Any field $\mathbb{F}$ satisfying the axioms so far listed is naturally called an *ordered field*. Of course, the set P is known as the set of *positive* elements of $\mathbb{F}$. Using Axiom 7(b), we see $\mathbb{F}$ is divided into three pairwise disjoint sets: P, $\{0\}$ and $\{-x : x \in P\}$. The latter of these is, of course, the set of *negative* elements from $\mathbb{F}$. The following definition introduces familiar notation for order.

Definition 2.4. We write $a < b$ or $b > a$, if $b - a \in P$. The meanings of $a \leq b$ and $b \geq a$ are now as expected.

Notice that $a > 0 \iff a = a - 0 \in P$ and $a < 0 \iff -a = 0 - a \in P$, so $a > 0$ and $a < 0$ agree with our usual notions of positive and negative.

Our goal is to capture all the properties of the real numbers with the axioms. The order axiom eliminates many fields from consideration. For example, Exercise 7 shows the field $\mathbb{Z}_3$ of Example 2.2 is not an ordered field. On the other hand, facts from elementary algebra imply $\mathbb{Q}$ is an ordered field. As noted above, $\mathbb{Q}$ does not contain all the real numbers, so the first seven axioms still don't characterize the real numbers.

Following are a few standard properties of ordered fields.

Theorem 2.5. *Let $\mathbb{F}$ be an ordered field and $a \in \mathbb{F}$. $a \neq 0$ iff $a^2 > 0$.*

Proof. ($\Rightarrow$) If $a > 0$, then $a^2 > 0$ by Axiom 7(a). If $a < 0$, then $-a > 0$ by Axiom 7(b) and $a^2 = 1a^2 = (-1)(-1)a^2 = (-a)^2 > 0$.

($\Leftarrow$) Since $0^2 = 0$, this is obvious. □

Theorem 2.6. *If $\mathbb{F}$ is an ordered field and $a, b, c \in \mathbb{F}$, then*

(a) $a < b \iff a + c < b + c$,

(b) $a < b \wedge b < c \implies a < c$,

(c) $a < b \wedge c > 0 \implies ac < bc$,

(d) $a < b \wedge c < 0 \implies ac > bc$.

Proof. (a) $a < b \iff b - a \in P \iff (b + c) - (a + c) \in P \iff a + c < b + c$.

(b) By supposition, both $b - a, c - b \in P$. Using the fact that P is closed under addition, we see $(b - a) + (c - b) = c - a \in P$. Therefore, $c > a$.

(c) Since both $b-a, c \in P$ and P is closed under multiplication, $c(b-a) = cb - ca \in P$ and, therefore, $ac < bc$.

(d) By assumption, $b-a, -c \in P$. Apply part (c) and Exercise 1. □

Theorem 2.7 (Two Out of Three Rule). *Let $\mathbb{F}$ be an ordered field and $a, b, c \in \mathbb{F}$. If $ab = c$ and any two of a, b or c are positive, then so is the third.*

Proof. If $a > 0$ and $b > 0$, then Axiom 7(a) implies $c > 0$. Next, suppose $a > 0$ and $c > 0$. In order to force a contradiction, suppose $b \leq 0$. In this case, Axiom 7(b) shows

$$0 \leq a(-b) = -(ab) = -c < 0,$$

which is impossible. □

Corollary 2.8. $1 > 0$

Proof. Exercise 2. □

Corollary 2.9. *Let $\mathbb{F}$ be an ordered field and $a \in \mathbb{F}$. If $a > 0$, then $a^{-1} > 0$. If $a < 0$, then $a^{-1} < 0$.*

Proof. Exercise 3. □

An ordered field begins to look like what we expect for the real numbers. The number line works pretty much as usual. Combining Corollary 2.8 and Axiom 7(a), it follows that $2 = 1 + 1 > 1 > 0$, $3 = 2 + 1 > 2 > 0$ and so forth. By induction, it is seen there is a copy of $\mathbb{N}$ embedded in $\mathbb{F}$. Similarly, there are also copies of $\mathbb{Z}$ and $\mathbb{Q}$ in $\mathbb{F}$. This shows every ordered field is infinite. But, there might be holes in the line. For example if $\mathbb{F} = \mathbb{Q}$, numbers like $\sqrt{2}$, e and π are missing.

Definition 2.10. If $\mathbb{F}$ is an ordered field and $a < b$ in $\mathbb{F}$, then $(a, b) = \{x \in \mathbb{F} : a < x < b\}$, $(a, \infty) = \{x \in \mathbb{F} : a < x\}$ and $(-\infty, a) = \{x \in \mathbb{F} : a > x\}$ are called open intervals. (The latter two are sometimes called open right and left rays, respectively.)

The sets $[a, b] = \{x \in \mathbb{F} : a \leq x \leq b\}$, $[a, \infty) = \{x \in \mathbb{F} : a \leq x\}$ and $(-\infty, a] = \{x \in \mathbb{F} : a \geq x\}$ are called closed intervals. (As above, the latter two are sometimes called closed rays.)

$[a, b) = \{x \in \mathbb{F} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{F} : a < x \leq b\}$ are half-open intervals.

The difference between the open and closed intervals is that open intervals don't contain their endpoints and closed intervals contain their endpoints. In the case of a ray, the interval only has one endpoint. It is incorrect to write a ray as $(a, \infty]$ or $[-\infty, a]$ because neither ∞ nor $-\infty$ is an element of $\mathbb{F}$. The symbols ∞ and $-\infty$ are just place holders telling us the intervals continue forever to the right or left.

2.2.1 Metric Properties

The order axiom on a field $\mathbb{F}$ allows us to introduce the idea of the distance between points in $\mathbb{F}$. To do this, we begin with the following familiar definition.

Definition 2.11. Let $\mathbb{F}$ be an ordered field. The *absolute value function* on $\mathbb{F}$ is a function $|\cdot| : \mathbb{F} \to \mathbb{F}$ defined as

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}.$$

The most important properties of the absolute value function are contained in the following theorem.

Theorem 2.12. *Let $\mathbb{F}$ be an ordered field and $x, y \in \mathbb{F}$. Then*

(a) $|x| \geq 0$ *and* $|x| = 0 \iff x = 0$;

(b) $|x| = |-x|$;

(c) $-|x| \leq x \leq |x|$;

(d) $|x| \leq y \iff -y \leq x \leq y$; *and,*

(e) $|x + y| \leq |x| + |y|$.

Proof. (a) The fact that $|x| \geq 0$ for all $x \in \mathbb{F}$ follows from Axiom 7(b). Since $0 = -0$, the second part is clear.

(b) If $x \geq 0$, then $-x \leq 0$ so that $|-x| = -(-x) = x = |x|$. If $x < 0$, then $-x > 0$ and $|x| = -x = |-x|$.

(c) If $x \geq 0$, then $-|x| = -x \leq x = |x|$. If $x < 0$, then $-|x| = -(-x) = x < -x = |x|$.

(d) This is left as Exercise 4.

(e) Add the two sets of inequalities $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$ to see $-(|x| + |y|) \leq x + y \leq |x| + |y|$. Now apply (d). This is usually called the *triangle inequality.*

□

From studying analytic geometry and calculus, we are used to thinking of $|x - y|$ as the distance between the numbers x and y. This notion of a distance between two points of a set can be generalized.

Definition 2.13. Let S be a set and $d : S \times S \to \mathbb{F}$ satisfy

(a) for all $x, y \in S$, $d(x, y) \geq 0$ and $d(x, y) = 0 \iff x = y$,

(b) for all $x, y \in S$, $d(x, y) = d(y, x)$, and

(c) for all $x, y, z \in S$, $d(x, z) \leq d(x, y) + d(y, z)$.

Then the function d is a *metric* on S. The pair (S, d) is called a *metric space.*

A metric is a function which defines the distance between any two points of a set.

Example 2.3. Let S be a set and define $d : S \times S \to S$ by

$$d(x, y) = \begin{cases} 1, & x \neq y \\ 0, & x = y \end{cases}.$$

It can readily be verified that d is a metric on S. This simplest of all metrics is called the *discrete* metric and it can be defined on any set. It's not often useful.

Theorem 2.14. *If $\mathbb{F}$ is an ordered field, then $d(x, y) = |x - y|$ is a metric on $\mathbb{F}$.*

Proof. Use parts (a), (b) and (e) of Theorem 2.12. □

The metric on $\mathbb{F}$ derived from the absolute value function is called the *standard* metric on $\mathbb{F}$. There are other metrics sometimes defined for specialized purposes, but we won't have need of them. Metrics will be revisited in Chapter 9.

2.3 The Completeness Axiom

All the axioms given so far are obvious from beginning algebra, and, on the surface, it's not obvious they haven't captured all the properties of the real numbers. Since $\mathbb{Q}$ satisfies them all, the following theorem shows we're not yet done.

Theorem 2.15. *There is no $\alpha \in \mathbb{Q}$ such that $\alpha^2 = 2$.*

Proof. Assume to the contrary that there is $\alpha \in \mathbb{Q}$ with $\alpha^2 = 2$. Then there are $p, q \in \mathbb{N}$ such that $\alpha = p/q$ with p and q relatively prime. Now,

$$\left(\frac{p}{q}\right)^2 = 2 \implies p^2 = 2q^2 \tag{2.1}$$

shows p^2 is even. Since the square of an odd number is odd, p must be even; i. e., $p = 2r$ for some $r \in \mathbb{N}$. Substituting this into (2.1), shows $2r^2 = q^2$. The same argument as above establishes q is also even. This contradicts the assumption that p and q are relatively prime. Therefore, no such α exists. $\square$

Since we suspect $\sqrt{2}$ is a perfectly fine real number, there's still something missing from the list of axioms. Completeness is the missing idea.

The Completeness Axiom is somewhat more complicated than the previous axioms, and several definitions are needed in order to state it.

2.3.1 Bounded Sets

Definition 2.16. A subset S of an ordered field $\mathbb{F}$ is *bounded above*, if there exists $M \in \mathbb{F}$ such that $M \geq x$ for all $x \in S$, and it is *bounded below*, if there exists $m \in \mathbb{F}$ such that $m \leq x$ for all $x \in S$. The elements M and m are called *upper* and *lower* bounds for S, respectively. If S is bounded both above and below, it is a simply called a *bounded* set.

There is no requirement in the definition that the upper and lower bounds for a set are elements of the set. They can be elements of the set, but typically are not. For example, if $S = (-\infty, 0)$, then $[0, \infty)$ is the set of all upper bounds for S, but none of them is in S. On the other hand, if $T = (-\infty, 0]$, then $[0, \infty)$ is again the set of all upper bounds for T, but in this case 0 is an upper bound which is also an element of T.

A set need not have upper or lower bounds. For example $S = (-\infty, 0)$ has no lower bounds, while $P = (0, \infty)$ has no upper bounds. The integers, $\mathbb{Z}$, has neither upper nor lower bounds. If a set has no upper bound, it is *unbounded above* and, if it has no lower bound, then it is *unbounded below*. In either case, it is usually just said to be *unbounded*.

If M is an upper bound for the set S, then every $x \geq M$ is also an upper bound for S. Considering some simple examples should lead you to suspect that among the upper bounds for a set, there is one that is best in the sense that everything greater is an upper bound and everything less is not an upper bound. This is the basic idea of completeness.

Definition 2.17. Suppose $\mathbb{F}$ is an ordered field and S is bounded above in $\mathbb{F}$. A number $B \in \mathbb{F}$ is called a *least upper bound* of S if

(a) B is an upper bound for S, and

(b) if α is any upper bound for S, then $B \leq \alpha$.

If S is bounded below in $\mathbb{F}$, then a number $b \in \mathbb{F}$ is called a *greatest lower bound* of S if

(a) b is a lower bound for S, and

(b) if α is any lower bound for S, then $\alpha \leq b$.

Theorem 2.18. *If $\mathbb{F}$ is an ordered field and $A \subset \mathbb{F}$ is nonempty, then A has at most one least upper bound and at most one greatest lower bound.*

Proof. Suppose u_1 and u_2 are both least upper bounds for A. Since u_1 and u_2 are both upper bounds for A, two applications of Definition 2.17 shows $u_1 \leq u_2 \leq u_1 \implies u_1 = u_2$. The proof of the other case is similar. $\square$

Definition 2.19. If $A \subset \mathbb{F}$ is nonempty and bounded above, then the least upper bound of A is written $\operatorname{lub} A$. When A is not bounded above, we write $\operatorname{lub} A = \infty$. When $A = \emptyset$, then $\operatorname{lub} A = -\infty$.

If $A \subset \mathbb{F}$ is nonempty and bounded below, then the greatest lower bound of A is written $\operatorname{glb} A$. When A is not bounded below, we write $\operatorname{glb} A = -\infty$. When $A = \emptyset$, then $\operatorname{glb} A = \infty$.[3]

Notice the symbol "∞" is not an element of $\mathbb{F}$. Writing $\operatorname{lub} A = \infty$ is just a convenient way to say A has no upper bounds. Similarly $\operatorname{lub} \emptyset = -\infty$ tells us $\emptyset$ has every real number as an upper bound.

Theorem 2.20. *Let $A \subset \mathbb{F}$ and $\alpha \in \mathbb{F}$. $\alpha = \operatorname{lub} A$ iff $(\alpha, \infty) \cap A = \emptyset$ and for all $\beta < \alpha$, $(\beta, \alpha] \cap A \neq \emptyset$. Similarly, $\alpha = \operatorname{glb} A$ iff $(-\infty, \alpha) \cap A = \emptyset$ and for all $\beta > \alpha$, $[\alpha, \beta) \cap A \neq \emptyset$.*

Proof. We will prove the first statement, concerning the least upper bound. The second statement, concerning the greatest lower bound, follows similarly.

($\Rightarrow$) If $x \in (\alpha, \infty) \cap A$, then α cannot be an upper bound of A, which is a contradiction. If there is an $\beta < \alpha$ such that $(\beta, \alpha] \cap A = \emptyset$, then from above, we conclude

$$\emptyset = ((\beta, \alpha] \cap A) \cup ((\alpha, \infty) \cap A) = (\beta, \infty) \cap A.$$

[3] Some people prefer the notation $\sup A$ and $\inf A$ instead of $\operatorname{lub} A$ and $\operatorname{glb} A$, respectively. They stand for the supremum and infimum of A.

So, $(\alpha+\beta)/2$ is an upper bound for A which is less than $\alpha = \operatorname{lub} A$. This contradiction shows $(\beta,\alpha]\cap A \neq \emptyset$.

($\Leftarrow$) The assumption that $(\alpha,\infty)\cap A = \emptyset$ implies $\alpha \geq \operatorname{lub} A$. On the other hand, suppose $\operatorname{lub} A < \alpha$. By assumption, there is an $x \in (\operatorname{lub} A, \alpha]\cap A$. This is clearly a contradiction, since $\operatorname{lub} A < x \in A$. Therefore, $\alpha = \operatorname{lub} A$. □

An eagle-eyed reader may wonder why the intervals in Theorem 2.20 are $(\beta,\alpha]$ and $[\alpha,\beta)$ instead of (β,α) and (α,β). Just consider the case $A = \{\alpha\}$ to see that the theorem fails when the intervals are open. When $\operatorname{lub} A \notin A$ or $\operatorname{glb} A \notin A$, the intervals can be open, as shown in the following corollary.

Corollary 2.21. *If A is bounded above and $\alpha = \operatorname{lub} A \notin A$, then for all $\beta < \alpha$, $(\beta,\alpha)\cap A$ is an infinite set. Similarly, if A is bounded below and $\alpha = \operatorname{glb} A \notin A$, then for all $\beta > \alpha$, $(\alpha,\beta)\cap A$ is an infinite set.*

Proof. Let $\beta < \alpha$. According to Theorem 2.20, there is an $x_1 \in (\beta,\alpha]\cap A$. By assumption, $x_1 < \alpha$. We continue by induction. Suppose $n \in \mathbb{N}$ and x_n has been chosen to satisfy $x_n \in (\beta,\alpha)\cap A$. Using Theorem 2.20 as before to choose $x_{n+1} \in (x_n,\alpha)\cap A$. The set $\{x_n : n \in \mathbb{N}\}$ is infinite and contained in $(\alpha-\varepsilon,\alpha)\cap A$.

The other statement in the corollary has a similar proof. □

When $\mathbb{F} = \mathbb{Q}$, Theorem 2.15 shows there is no least upper bound for $A = \{x : x^2 < 2\}$ in $\mathbb{Q}$. It seems $\mathbb{Q}$ has a hole where this least upper bound should be. Adding the following completeness axiom enlarges $\mathbb{Q}$ to fill in the holes.

Axiom 8 (Completeness). Every nonempty set which is bounded above has a least upper bound.

This is the final axiom. Any field $\mathbb{F}$ satisfying all eight axioms is called a *complete ordered field*. We assume the existence of a complete ordered field, $\mathbb{R}$, called the *real numbers.*

In naive set theory it can be shown that if $\mathbb{F}_1$ and $\mathbb{F}_2$ are both complete ordered fields, then they are the same, in the following sense. There exists a unique bijective function $i : \mathbb{F}_1 \to \mathbb{F}_2$ such that $i(a+b) = i(a)+i(b)$, $i(ab) = i(a)i(b)$ and $a < b \iff i(a) < i(b)$. Such a function i is called an *order isomorphism*. The existence of such an order isomorphism shows that $\mathbb{R}$ is essentially unique. More reading on this topic can be done in some advanced texts [13, 14].

Every statement about upper bounds has a dual statement about lower bounds. A proof of the following dual to Axiom 8 is left as an exercise.

Corollary 2.22. *Every nonempty subset of $\mathbb{R}$ which is bounded below has a greatest lower bound.*

In Section 2.4 it will be proved that there is an $x \in \mathbb{R}$ satisfying $x^2 = 2$. This will show $\mathbb{R}$ removes the deficiency of $\mathbb{Q}$ highlighted by Theorem 2.15. The Completeness Axiom plugs up the holes in $\mathbb{Q}$.

2.3.2 Some Consequences of Completeness

The property of completeness is what separates analysis from geometry and algebra. Analysis requires the use of approximation, infinity and more dynamic visualizations than algebra or classical geometry. The rest of this course is largely concerned with applications of completeness.

Theorem 2.23 (Archimedean Principle). *If $a \in \mathbb{R}$, then there exists $n_a \in \mathbb{N}$ such that $n_a > a$.*

Proof. If the theorem is false, then a is an upper bound for $\mathbb{N}$. Let $\beta = \operatorname{lub} \mathbb{N}$. According to Theorem 2.20 there is an $m \in \mathbb{N}$ such that $m > \beta - 1$. But, this is a contradiction because $\beta = \operatorname{lub} \mathbb{N} < m + 1 \in \mathbb{N}$. □

Some other variations on this theme are in the following corollaries.

Corollary 2.24. *Let $a, b \in \mathbb{R}$ with $a > 0$.*

(a) *There is an $n \in \mathbb{N}$ such that $an > b$.*

(b) *There is an $n \in \mathbb{N}$ such that $0 < 1/n < a$.*

(c) *There is an $n \in \mathbb{N}$ such that $n - 1 \leq a < n$.*

Proof. (a) Use Theorem 2.23 to find $n \in \mathbb{N}$ where $n > b/a$.

(b) Let $b = 1$ in part (a).

(c) Theorem 2.23 guarantees that $S = \{n \in \mathbb{N} : n > a\} \neq \emptyset$. If n is the least element of this set, then $n - 1 \notin S$ and $n - 1 \leq a < n$. □

Corollary 2.25. *If I is any interval from $\mathbb{R}$, then $I \cap \mathbb{Q} \neq \emptyset$ and $I \cap \mathbb{Q}^c \neq \emptyset$.*

Proof. See Exercises 15 and 17. □

A subset of $\mathbb{R}$ which intersects every interval is said to be *dense* in $\mathbb{R}$. Corollary 2.25 shows both the rational and irrational numbers are dense.

2.4 Comparisons of $\mathbb{Q}$ and $\mathbb{R}$

All of the above still does not establish that $\mathbb{Q}$ is different from $\mathbb{R}$. In Theorem 2.15, it was shown that the equation $x^2 = 2$ has no solution in $\mathbb{Q}$. The following theorem shows $x^2 = 2$ does have solutions in $\mathbb{R}$. Since a copy of $\mathbb{Q}$ is embedded in $\mathbb{R}$, it follows, in a sense, that $\mathbb{R}$ is bigger than $\mathbb{Q}$.

Theorem 2.26. *There is a positive $\alpha \in \mathbb{R}$ such that $\alpha^2 = 2$.*

Proof. Let $S = \{x > 0 : x^2 < 2\}$. Then $1 \in S$, so $S \neq \emptyset$. If $x \geq 2$, then Theorem 2.6(c) implies $x^2 \geq 4 > 2$, so S is bounded above. The Completeness Axiom gives the existence of $\alpha = \operatorname{lub} S > 1$. It will be shown that $\alpha^2 = 2$.

Suppose first that $\alpha^2 < 2$. This assumption implies $(2-\alpha^2)/(2\alpha+1) > 0$. According to Corollary 2.24, there is an $n \in \mathbb{N}$ large enough so

$$0 < \frac{1}{n} < \frac{2-\alpha^2}{2\alpha+1} \iff 0 < \frac{2\alpha+1}{n} < 2-\alpha^2.$$

Therefore,

$$\begin{aligned}\left(\alpha+\frac{1}{n}\right)^2 &= \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} = \alpha^2 + \frac{1}{n}\left(2\alpha + \frac{1}{n}\right)\\ &< \alpha^2 + \frac{2\alpha+1}{n} < \alpha^2 + (2-\alpha^2) = 2\end{aligned}$$

contradicts the fact that $\alpha = \operatorname{lub} S$. Therefore, $\alpha^2 \geq 2$.

Next, assume $\alpha^2 > 2$. In this case, choose $n \in \mathbb{N}$ so

$$0 < \frac{1}{n} < \frac{\alpha^2-2}{2\alpha} \iff 0 < \frac{2\alpha}{n} < \alpha^2 - 2.$$

Then

$$\left(\alpha-\frac{1}{n}\right)^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n} > \alpha^2 - (\alpha^2-2) = 2,$$

again contradicts that $\alpha = \operatorname{lub} S$.

Therefore, $\alpha^2 = 2$. □

Theorem 2.15 leads to the obvious question of how much bigger $\mathbb{R}$ is than $\mathbb{Q}$. First, note that since $\mathbb{N} \subset \mathbb{Q}$, it is clear that $\operatorname{card}(\mathbb{Q}) \geq \aleph_0$. On the other hand, every $q \in \mathbb{Q}$ has a unique reduced fractional representation $q = m(q)/n(q)$ with $m(q) \in \mathbb{Z}$ and $n(q) \in \mathbb{N}$. This gives an injective function $f : \mathbb{Q} \to \mathbb{Z} \times \mathbb{N}$ defined by $f(q) = (m(q), n(q))$, and according to Theorem 1.20, $\operatorname{card}(\mathbb{Q}) \leq \operatorname{card}(\mathbb{Z} \times \mathbb{N}) = \aleph_0$. The following theorem ensues.

$$\begin{array}{lllllll}
\alpha_1 = & .\alpha_1(1) & \alpha_1(2) & \alpha_1(3) & \alpha_1(4) & \alpha_1(5) & \ldots \\
\alpha_2 = & .\alpha_2(1) & \alpha_2(2) & \alpha_2(3) & \alpha_2(4) & \alpha_2(5) & \ldots \\
\alpha_3 = & .\alpha_3(1) & \alpha_3(2) & \alpha_3(3) & \alpha_3(4) & \alpha_3(5) & \ldots \\
\alpha_4 = & .\alpha_4(1) & \alpha_4(2) & \alpha_4(3) & \alpha_4(4) & \alpha_4(5) & \ldots \\
\alpha_5 = & .\alpha_5(1) & \alpha_5(2) & \alpha_5(3) & \alpha_5(4) & \alpha_5(5) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{array}$$

Figure 2.1: The proof of Theorem 2.28 is called the "diagonal argument" because it constructs a new number z by working down the main diagonal of the array shown above, making sure $z(n) \neq \alpha_n(n)$ for each $n \in \mathbb{N}$.

Theorem 2.27. $\operatorname{card}(\mathbb{Q}) = \aleph_0$.

In 1874, Georg Cantor first showed that $\mathbb{R}$ is not countable. The following proof is his famous diagonal argument from 1891.

Theorem 2.28. $\operatorname{card}(\mathbb{R}) > \aleph_0$.

Proof. It suffices to prove that $\operatorname{card}([0,1]) > \aleph_0$. If this is not true, then there is a bijection $\alpha : \mathbb{N} \to [0,1]$; i.e.,

$$[0,1] = \{\alpha_n : n \in \mathbb{N}\}. \tag{2.2}$$

Each $x \in [0,1]$ can be written in the decimal form $x = \sum_{n=1}^{\infty} x(n)/10^n$ where $x(n) \in \{0,1,2,3,4,5,6,7,8,9\}$ for each $n \in \mathbb{N}$. This decimal representation is not necessarily unique. For example,

$$\frac{1}{2} = \frac{5}{10} = \frac{4}{10} + \sum_{n=2}^{\infty} \frac{9}{10^n}.$$

In such a case, there is a choice of $x(n)$ so it is constantly 9 or constantly 0 from some N onward. When given a choice, we will always opt to end the number with a string of nines. With this convention, the decimal representation of x is unique. (Prove this!)

Define $z \in [0,1]$ by choosing $z(n) \in \{0,1\}$ such that $z(n) \neq \alpha_n(n)$. Let $z = \sum_{n=1}^{\infty} z(n)/10^n$. Since $z \in [0,1]$, there is an $n \in \mathbb{N}$ such that $z = \alpha_n$. But, this is impossible because $z(n)$ differs from α_n in the nth decimal place. This contradiction shows $\operatorname{card}([0,1]) > \aleph_0$. $\square$

Around the turn of the twentieth century these then-new ideas about infinite sets were very controversial in mathematics. This is because some of these ideas are very unintuitive. For example, the rational numbers are a countable set and the irrational numbers are uncountable, yet between every two rational numbers is an uncountable number of irrational numbers and between every two irrational numbers there is a countably infinite number of rational numbers. It would seem there are either too few or too many gaps in the sets to make this possible. Such a seemingly paradoxical situation flies in the face of our intuition, which was developed with finite sets in mind.

This brings us back to the discussion of cardinalities and the Continuum Hypothesis at the end of Section 1.1.5. Most of the time, people working in real analysis assume the Continuum Hypothesis is true. With this assumption and Theorem 2.28 it follows that whenever $A \subset \mathbb{R}$, then either $\operatorname{card}(A) \leq \aleph_0$ or $\operatorname{card}(A) = \operatorname{card}(\mathbb{R}) = \operatorname{card}(\mathcal{P}(\mathbb{N}))$.[4] Since $\mathcal{P}(\mathbb{N})$ has *many* more elements than $\mathbb{N}$, any countable subset of $\mathbb{R}$ is considered to be a small set, in the sense of cardinality, even if it is infinite. This works against the intuition of many beginning students who are not used to thinking of $\mathbb{Q}$, or any other infinite set as being small. But it turns out to be quite useful because the fact that the union of a countably infinite number of countable sets is still countable can be exploited in many ways.[5]

In later chapters, other useful small versus large dichotomies will be found.

2.5 Exercises

Exercise 2.1. Prove that if $a, b \in \mathbb{F}$, where $\mathbb{F}$ is a field, then $(-a)b = -(ab) = a(-b)$.

Exercise 2.2. Prove $1 > 0$.

Exercise 2.3. Prove Corollary 2.9: If $a > 0$, then so is a^{-1}. If $a < 0$, then so is a^{-1}.

Exercise 2.4. Prove $|x| \leq y$ iff $-y \leq x \leq y$.

Exercise 2.5. Let $\mathbb{F}$ be an ordered field and $a, b, c \in \mathbb{F}$. If $ab = c$ and two of a, b and c are negative, then the third is positive.

---

[4] Since $\aleph_0$ is the smallest infinite cardinal, $\aleph_1$ is used to denote the smallest uncountable cardinal. You will also see $\operatorname{card}(\mathbb{R}) = \mathfrak{c}$, where $\mathfrak{c}$ is the old-style, (Fraktur) German letter c, standing for the "cardinality of the continuum." Assuming the Continuum Hypothesis, it follows that $\aleph_0 < \aleph_1 = \mathfrak{c}$.

[5] See Problem 25 on page 18.

Exercise 2.6. If $S \subset \mathbb{R}$ is bounded above, then

$$\operatorname{lub} S = \operatorname{glb}\{x : x \text{ is an upper bound for } S\}.$$

Exercise 2.7. Prove there is no set $P \subset \mathbb{Z}_3$ which makes $\mathbb{Z}_3$ into an ordered field.

Exercise 2.8. If α is an upper bound for S and $\alpha \in S$, then $\alpha = \operatorname{lub} S$.

Exercise 2.9. Let A and B be subsets of $\mathbb{R}$ that are bounded above. Define $A + B = \{a + b : a \in A \wedge b \in B\}$. Prove that $\operatorname{lub}(A + B) = \operatorname{lub} A + \operatorname{lub} B$.

Exercise 2.10. If $A \subset \mathbb{Z}$ is bounded below, then A has a least element.

Exercise 2.11. If $\mathbb{F}$ is an ordered field and $a \in \mathbb{F}$ such that $0 \le a < \varepsilon$ for every $\varepsilon > 0$, then $a = 0$.

Exercise 2.12. Let $x \in \mathbb{R}$. Prove $|x| < \varepsilon$ for all $\varepsilon > 0$ iff $x = 0$.

Exercise 2.13. If p is a prime number, then the equation $x^2 = p$ has no rational solutions.

Exercise 2.14. If p is a prime number and $\varepsilon > 0$, then there are $x, y \in \mathbb{Q}$ such that $x^2 < p < y^2 < x^2 + \varepsilon$.

Exercise 2.15. If $a < b$, then $(a, b) \cap \mathbb{Q} \neq \emptyset$.

Exercise 2.16. If $q \in \mathbb{Q}$ and $a \in \mathbb{R} \setminus \mathbb{Q}$, then $q + a \in \mathbb{R} \setminus \mathbb{Q}$. Moreover, if $q \neq 0$, then $aq \in \mathbb{R} \setminus \mathbb{Q}$.

Exercise 2.17. Prove that if $a < b$, then there is a $q \in \mathbb{Q}$ such that $a < \sqrt{2}q < b$.

Exercise 2.18. If $\mathbb{F}$ is an ordered field and $x_1, x_2, \ldots, x_n \in \mathbb{F}$ for some $n \in \mathbb{N}$, then

$$\left|\sum_{i=1}^{n} x_i\right| \le \sum_{i=1}^{n} |x_i|. \tag{2.6}$$

Exercise 2.19. Let $\mathbb{F}$ be an ordered field. (a) Prove $\mathbb{F}$ has no upper or lower bounds.

(b) Every element of $\mathbb{F}$ is both an upper and lower bound for $\emptyset$.

Exercise 2.20. Prove Corollary 2.22.

Exercise 2.21. Prove $\operatorname{card}(\mathbb{Q}^c) = \mathfrak{c}$.

Exercise 2.22. If $A \subset \mathbb{R}$ and $B = \{x : x \text{ is an upper bound for } A\}$, then $\operatorname{lub}(A) = \operatorname{glb}(B)$.

Chapter 3

Sequences

We begin our study of analysis with sequences. There are several reasons for starting here. First, sequences are the simplest way to introduce limits, the central idea of calculus. Second, sequences are a direct route to the topology of the real numbers. The combination of limits and topology provides the tools to finally prove the theorems you've already used in your calculus courses.

3.1 Basic Properties

Definition 3.1. A *sequence* is a function $a : \mathbb{N} \to \mathbb{R}$.

Instead of using the standard function notation of $a(n)$ for sequences, it is usually more convenient to write the argument of the function as a subscript, a_n.

Example 3.1. Let the sequence $a_n = 1 - 1/n$. The first few elements are $a_1 = 0,\ a_2 = 1/2,\ a_3 = 2/3$, etc.

Example 3.2. Let the sequence $b_n = 2^n$. Then $b_1 = 2,\ b_2 = 4,\ b_3 = 8$, etc.

Example 3.3. Let the sequence $c_n = 100 - 5n$ so $c_1 = 95$, $c_2 = 90$, $c_3 = 85$, etc.

Example 3.4. If a and r are nonzero constants, then a sequence given by $c_1 = a$, $c_2 = ar$, $c_3 = ar^2$ and in general $c_n = ar^{n-1}$ is called a *geometric sequence.* The number r is called the *ratio* of the sequence. A geometric sequence can always be recognized by noticing that $\frac{c_{n+1}}{c_n} = r$ for all $n \in \mathbb{N}$. Example 3.2 is a geometric sequence with $a = r = 2$.

Example 3.5. If a and d are constants with $d \neq 0$, then a sequence of the form $d_n = a + (n-1)d$ is called an *arithmetic sequence.* Another way of looking at this is that d_n is an arithmetic sequence if $d_{n+1} - d_n = d$ for all $n \in \mathbb{N}$. Example 3.3 is an arithmetic sequence with $a = 95$ and $d = -5$.

Example 3.6. Some sequences are not defined by an explicit formula, but are defined *recursively*. This is an inductive method of definition in which successive terms of the sequence are defined by using other terms of the sequence. The most famous of these is the *Fibonacci sequence.* To define the Fibonacci sequence, f_n, let $f_1 = 1$, $f_2 = 1$ and for $n > 2$, let $f_n = f_{n-2} + f_{n-1}$. The first few terms are $1, 1, 2, 3, 5, 8, 13, \ldots$. There actually is a simple formula that directly gives f_n, and its derivation is Exercise 14.

Example 3.7. These simple definitions can lead to complex problems. One famous case is a *hailstone sequence*. Let h_1 be any natural number. For $n > 1$, recursively define

$$h_n = \begin{cases} 3h_{n-1} + 1, & \text{if } h_{n-1} \text{ is odd} \\ h_{n-1}/2, & \text{if } h_{n-1} \text{ is even} \end{cases}.$$

Lothar Collatz conjectured in 1937 that any hailstone sequence eventually settles down to repeating the pattern $1, 4, 2, 1, 4, 2, \ldots$. Many people have tried to prove this and all have failed.

It's often inconvenient for the domain of a sequence to be $\mathbb{N}$, as required by Definition 3.1. For example, the sequence beginning $1, 2, 4, 8, \ldots$ can be written $2^0, 2^1, 2^2, 2^3, \ldots$. Written this way, it's natural to let the sequence function be 2^n with domain ω. As long as there is a simple substitution to write the sequence function in the form of Definition 3.1, there's no reason to adhere to the letter of the law. In general, the domain of a sequence can be any set of the form $\{n \in \mathbb{Z} : n \geq N\}$ for some $N \in \mathbb{Z}$.

Definition 3.2. A sequence a_n is *bounded* if $\{a_n : n \in \mathbb{N}\}$ is a bounded set. This definition is extended in the obvious way to *bounded above* and *bounded below*.

The sequence of Example 3.1 is bounded, but the sequence of Example 3.2 is not, although it is bounded below.

Definition 3.3. A sequence a_n *converges* to $L \in \mathbb{R}$ if for all $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$, then $|a_n - L| < \varepsilon$. If a sequence does not converge, then it is said to *diverge*.

When a_n converges to L, we write $\lim_{n\to\infty} a_n = L$, or often, more simply, $a_n \to L$.

Example 3.8. Let $a_n = 1 - 1/n$ be as in Example 3.1. We claim $a_n \to 1$. To see this, let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $1/N < \varepsilon$. If $n \geq N$

$$|a_n - 1| = |(1 - 1/n) - 1| = 1/n \leq 1/N < \varepsilon,$$

so $a_n \to 1$.

Example 3.9. The sequence $b_n = 2^n$ of Example 3.2 diverges. To see this, suppose not. Then there is an $L \in \mathbb{R}$ such that $b_n \to L$. If $\varepsilon = 1$, there must be an $N \in \mathbb{N}$ such that $|b_n - L| < \varepsilon$ whenever $n \geq N$. Choose $n \geq N$. $|L - 2^n| < 1$ implies $L < 2^n + 1$. But, then

$$b_{n+1} - L = 2^{n+1} - L > 2^{n+1} - (2^n + 1) = 2^n - 1 \geq 1 = \varepsilon.$$

This violates the condition on N. We conclude that for every $L \in \mathbb{R}$ there exists an $\varepsilon > 0$ such that for no $N \in \mathbb{N}$ is it true that whenever $n \geq N$, then $|b_n - L| < \varepsilon$. Therefore, b_n diverges.

Definition 3.4. A sequence a_n *diverges to* ∞ if for every $B > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n > B$. The sequence a_n is said to *diverge to* $-\infty$ if $-a_n$ diverges to ∞.

When a_n diverges to ∞, we write $\lim_{n\to\infty} a_n = \infty$, or often, more simply, $a_n \to \infty$.

A common mistake is to forget that $a_n \to \infty$ actually means the sequence *diverges* in a particular way. Don't be fooled by the suggestive notation into treating ∞ as a number!

Example 3.10. It is easy to prove that the sequence $a_n = 2^n$ of Example 3.2 diverges to ∞.

Theorem 3.5. *If $a_n \to L$, then L is unique.*

Proof. Suppose $a_n \to L_1$ and $a_n \to L_2$. Let $\varepsilon > 0$. According to Definition 3.2, there exist $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - L_1| < \varepsilon/2$ and $n \geq N_2$ implies $|a_n - L_2| < \varepsilon/2$. Set $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Since ε is an arbitrary positive number an application of Exercise 12 shows $L_1 = L_2$. $\square$

Theorem 3.6. *$a_n \to L$ iff for all $\varepsilon > 0$, the set $\{n : a_n \notin (L - \varepsilon, L + \varepsilon)\}$ is finite.*

Proof. ($\Rightarrow$) Let $\varepsilon > 0$. According to Definition 3.2, there is an $N \in \mathbb{N}$ such that $\{a_n : n \geq N\} \subset (L-\varepsilon, L+\varepsilon)$. Then $\{n : a_n \notin (L-\varepsilon, L+\varepsilon)\} \subset \{1, 2, \ldots, N-1\}$, which is finite.

($\Leftarrow$) Let $\varepsilon > 0$. By assumption $\{n : a_n \notin (L-\varepsilon, L+\varepsilon)\}$ is finite, so let $N = \max\{n : a_n \notin (L-\varepsilon, L+\varepsilon)\} + 1$. If $n \geq N$, then $a_n \in (L-\varepsilon, L+\varepsilon)$. By Definition 3.2, $a_n \to L$. □

Corollary 3.7. *If a_n converges, then a_n is bounded.*

Proof. Suppose $a_n \to L$. According to Theorem 3.6 there are a finite number of terms of the sequence lying outside $(L-1, L+1)$. Since any finite set is bounded, the conclusion follows. □

The converse of this theorem is not true. For example, $a_n = (-1)^n$ is bounded, but does not converge. The main use of Corollary 3.7 is as a quick first check to see whether a sequence might converge. It's usually pretty easy to determine whether a sequence is bounded. If it isn't, it must diverge.

The following theorem lets us analyze some complicated sequences by breaking them down into combinations of simpler sequences.

Theorem 3.8. *Let a_n and b_n be sequences such that $a_n \to A$ and $b_n \to B$. Then*

(a) $a_n + b_n \to A + B$,

(b) $a_n b_n \to AB$, *and*

(c) $a_n / b_n \to A/B$ *as long as* $b_n \neq 0$ *for all* $n \in \mathbb{N}$ *and* $B \neq 0$.

Proof. (a) Let $\varepsilon > 0$. There are $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1$ implies $|a_n - A| < \varepsilon/2$ and $n \geq N_2$ implies $|b_n - B| < \varepsilon/2$. Define $N = \max\{N_1, N_2\}$. If $n \geq N$, then

$$|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore $a_n + b_n \to A + B$.

(b) Let $\varepsilon > 0$ and $\alpha > 0$ be an upper bound for $|a_n|$. Choose $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |a_n - A| < \varepsilon/2(|B|+1)$ and $n \geq N_2 \implies |b_n - B| <$

$\varepsilon/2\alpha$. If $n \geq N = \max\{N_1, N_2\}$, then

$$\begin{aligned}|a_n b_n - AB| &= |a_n b_n - a_n B + a_n B - AB| \\ &\leq |a_n b_n - a_n B| + |a_n B - AB| \\ &= |a_n||b_n - B| + |B||a_n - A| \\ &< \alpha \frac{\varepsilon}{2\alpha} + |B| \frac{\varepsilon}{2(|B|+1)} \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon.\end{aligned}$$

(c) First, notice that it suffices to show that $1/b_n \to 1/B$, because part (b) of this theorem can be used to achieve the full result.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that the following two conditions are satisfied: $n \geq N \implies |b_n| > |B|/2$ and $|b_n - B| < B^2\varepsilon/2$. Then, when $n \geq N$,

$$\left|\frac{1}{b_n} - \frac{1}{B}\right| = \left|\frac{B - b_n}{b_n B}\right| < \left|\frac{B^2\varepsilon/2}{(B/2)B}\right| = \varepsilon.$$

Therefore $1/b_n \to 1/B$.

□

If you're not careful, you can easily read too much into the previous theorem and try to use its converse. Consider the sequences $a_n = (-1)^n$ and $b_n = -a_n$. Their sum, $a_n + b_n = 0$, product $a_n b_n = -1$ and quotient $a_n/b_n = -1$ all converge, but the original sequences diverge.

It is often easier to prove that a sequence converges by comparing it with a known sequence than it is to analyze it directly. For example, a sequence such as $a_n = \sin^2 n/n^3$ can easily be seen to converge to 0 because it is dominated by $1/n^3$. The following theorem makes this idea more precise. It's called the Sandwich Theorem here, but is also called the Squeeze, Pinching, Pliers or Comparison Theorem in different texts.

Theorem 3.9 (Sandwich Theorem)**.** *Suppose a_n, b_n and c_n are sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$.*

(a) *If $a_n \to L$ and $c_n \to L$, then $b_n \to L$.*

(b) *If $b_n \to \infty$, then $c_n \to \infty$.*

(c) *If $b_n \to -\infty$, then $a_n \to -\infty$.*

Proof. (a) Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ large enough so that when $n \geq N$, then $L - \varepsilon < a_n$ and $c_n < L + \varepsilon$. These inequalities imply $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$. Theorem 3.6 shows $b_n \to L$.

(b) Let $B > 0$ and choose $N \in \mathbb{N}$ so that $n \geq N \implies b_n > B$. Then $c_n \geq b_n > B$ whenever $n \geq N$. This shows $c_n \to \infty$.

(c) This is essentially the same as part (b).

$\square$

3.2 Monotone Sequences

One of the problems with using the definition of convergence to prove a given sequence converges is the limit of the sequence must be known in order to verify that the sequence converges. This gives rise in the best cases to a "chicken and egg" problem of somehow determining the limit before you even know the sequence converges. In the worst case, there is no nice representation of the limit to use, so you don't even have a "target" to shoot at. The next few sections are ultimately concerned with removing this deficiency from Definition 3.2, but some interesting side-issues are explored along the way.

Not surprisingly, we begin with the simplest case.

Definition 3.10. A sequence a_n is *increasing*, if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. It is *strictly increasing* if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

A sequence a_n is *decreasing*, if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$. It is *strictly decreasing* if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

If a_n is any of the four types listed above, then it is said to be a *monotone* sequence.

Notice the $\leq$ and $\geq$ in the definitions of increasing and decreasing sequences, respectively. Many calculus texts use strict inequalities because they seem to better match the intuitive idea of what an increasing or decreasing sequence should do. For us, the non-strict inequalities are more convenient. A consequence of this is that a constant sequence is both increasing and decreasing.

Theorem 3.11. *A bounded monotone sequence converges.*

Proof. Suppose a_n is a bounded increasing sequence and $\varepsilon > 0$. The Completeness Axiom implies the existence of $L = \operatorname{lub}\{a_n : n \in \mathbb{N}\}$. Clearly, $a_n \leq L$ for all $n \in \mathbb{N}$. According to Theorem 2.20, there exists an $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$. Because the sequence is increasing, $L \geq a_n \geq a_N > L - \varepsilon$ for all $n \geq N$. This shows $a_n \to L$.

If a_n is decreasing, let $b_n = -a_n$ and apply the preceding argument. $\square$

The key idea of this proof is the existence of the least upper bound of the sequence when the range of the sequence is viewed as a set of numbers. This means the Completeness Axiom implies Theorem 3.11. In fact, it isn't hard to prove Theorem 3.11 also implies the Completeness Axiom, showing they are equivalent statements. Because of this, Theorem 3.11 is often used as the Completeness Axiom on $\mathbb{R}$ instead of the least upper bound property used in Axiom 8.

Example 3.11. The sequence $e_n = \left(1 + \frac{1}{n}\right)^n$ converges.

Looking at the first few terms of this sequence, $e_1 = 2$, $e_2 = 2.25$, $e_3 \approx 2.37$, $e_4 \approx 2.44$, it seems to be increasing. To show this is indeed the case, fix $n \in \mathbb{N}$ and use the binomial theorem to expand the product as

$$e_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} \tag{3.1}$$

and

$$e_{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(n+1)^k}. \tag{3.2}$$

For $1 \le k \le n$, the kth term of (3.1) is

$$\begin{aligned}
\binom{n}{k}\frac{1}{n^k} &= \frac{n(n-1)(n-2)\cdots(n-(k-1))}{k!n^k} \\
&= \frac{1}{k!}\frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-k+1}{n} \\
&= \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-1}{n}\right) \\
&< \frac{1}{k!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right)\cdots\left(1-\frac{k-1}{n+1}\right) \\
&= \frac{1}{k!}\left(\frac{n}{n+1}\right)\left(\frac{n-1}{n+1}\right)\cdots\left(\frac{n+1-(k-1)}{n+1}\right) \\
&= \frac{(n+1)n(n-1)(n-2)\cdots(n+1-(k-1))}{k!(n+1)^k} \\
&= \binom{n+1}{k}\frac{1}{(n+1)^k},
\end{aligned}$$

which is the kth term of (3.2). Since (3.2) also has one more positive term in the sum, it follows that $e_n < e_{n+1}$, and the sequence e_n is strictly increasing.

Noting that $1/k! \leq 1/2^{k-1}$ for $k \in \mathbb{N}$, we can bound the kth term of (3.1).

$$\begin{aligned}\binom{n}{k}\frac{1}{n^k} &= \frac{n!}{k!(n-k)!}\frac{1}{n^k} \\ &= \frac{n-1}{n}\frac{n-2}{n}\cdots\frac{n-k+1}{n}\frac{1}{k!} \\ &< \frac{1}{k!} \\ &\leq \frac{1}{2^{k-1}}.\end{aligned}$$

Substituting this into (3.1) yields

$$\begin{aligned}e_n &= \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &< 1+1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^{n-1}} \\ &= 1+\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}} < 3,\end{aligned}$$

so e_n is bounded.

Since e_n is increasing and bounded, Theorem 3.11 implies e_n converges. Of course, you probably remember from your calculus course that $e_n \to e \approx 2.71828$.

Theorem 3.12. *An unbounded monotone sequence diverges to ∞ or $-\infty$, depending on whether it is increasing or decreasing, respectively.*

Proof. Suppose a_n is increasing and unbounded. If $B > 0$, the fact that a_n is unbounded yields an $N \in \mathbb{N}$ such that $a_N > B$. Since a_n is increasing, $a_n \geq a_N > B$ for all $n \geq N$. This shows $a_n \to \infty$.

The proof when the sequence decreases is similar. □

3.3 The Bolzano-Weierstrass Theorem

Definition 3.13. Let a_n be a sequence and $\sigma : \mathbb{N} \to \mathbb{N}$ be a function such that $m < n$ implies $\sigma(m) < \sigma(n)$; i.e., $\sigma(n)$ is a strictly increasing sequence of natural numbers. Then $b_n = a \circ \sigma(n) = a_{\sigma(n)}$ is a *subsequence* of a_n.

The idea here is that the subsequence b_n is a new sequence formed from an old sequence a_n by possibly leaving terms out of a_n. In other words, all the terms of b_n must also appear in a_n, and they must appear in the same order.

Example 3.12. Let $\sigma(n) = 3n$ and a_n be a sequence. Then the subsequence $a_{\sigma(n)}$ looks like

$$a_3, a_6, a_9, \ldots, a_{3n}, \ldots$$

The subsequence has every third term of the original sequence.

Example 3.13. If $a_n = \sin(n\pi/2)$, then some possible subsequences are

$$b_n = a_{4n+1} \implies b_n = 1 \text{ for all } n,$$

$$c_n = a_{2n} \implies c_n = 0 \text{ for all } n,$$

and

$$d_n = a_{n^2} \implies d_n = (1 + (-1)^{n+1})/2.$$

Theorem 3.14. *$a_n \to L$ iff every subsequence of a_n converges to L.*

Proof. ($\Rightarrow$) Suppose $\sigma : \mathbb{N} \to \mathbb{N}$ is strictly increasing, as in the preceding definition. With a simple induction argument, it can be seen that $\sigma(n) \geq n$ for all n. (See Exercise 8.)

Now, suppose $a_n \to L$ and $b_n = a_{\sigma(n)}$ is a subsequence of a_n. If $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \in (L - \varepsilon, L + \varepsilon)$. From the preceding paragraph, it follows that when $n \geq N$, then $b_n = a_{\sigma(n)} = a_m$ for some $m \geq n$. So, $b_n \in (L - \varepsilon, L + \varepsilon)$ and $b_n \to L$.

($\Leftarrow$) Since a_n is a subsequence of itself, it is obvious that $a_n \to L$. $\square$

The main use of Theorem 3.14 is not to show that sequences converge, but, rather to show they diverge. It gives two strategies for doing this: find two subsequences converging to different limits, or find a divergent subsequence. In Example 3.13, the subsequences b_n and c_n demonstrate the first strategy, while d_n demonstrates the second.

Even if a given sequence is badly behaved, it is possible there are well-behaved subsequences. For example, consider the divergent sequence $a_n = (-1)^n$. In this case, a_n diverges, but the two subsequences $a_{2n} = 1$ and $a_{2n+1} = -1$ are constant sequences, so they converge.

Theorem 3.15. *Every sequence has a monotone subsequence.*

Proof. Let a_n be a sequence and $T = \{n \in \mathbb{N} : m > n \implies a_m \geq a_n\}$. There are two cases to consider, depending on whether T is finite.

First, assume T is infinite. Define $\sigma(1) = \min T$ and assuming $\sigma(n)$ is defined, set $\sigma(n+1) = \min T \setminus \{\sigma(1), \sigma(2), \ldots, \sigma(n)\}$. This inductively defines a strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$. The definition of T guarantees $a_{\sigma(n)}$ is an increasing subsequence of a_n.

Now, assume T is finite. Let $\sigma(1) = \max T + 1$. If $\sigma(n)$ has been chosen for some $n > \max T$, then the definition of T implies there is an $m > \sigma(n)$ such that $a_m \leq a_{\sigma(n)}$. Set $\sigma(n+1) = m$. This inductively defines the strictly increasing function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $a_{\sigma(n)}$ is a decreasing subsequence of a_n. □

If the sequence in Theorem 3.15 is bounded, then the corresponding monotone subsequence is also bounded. Recalling Theorem 3.11, we arrive at the following famous theorem.

Theorem 3.16 (Bolzano-Weierstrass[1]). *Every bounded sequence has a convergent subsequence.*

3.4 Lower and Upper Limits of a Sequence

There are an uncountable number of strictly increasing functions $\sigma : \mathbb{N} \to \mathbb{N}$, so every sequence a_n has an uncountable number of subsequences. If a_n converges, then Theorem 3.14 shows all of these subsequences converge to the same limit. It's also apparent that when $a_n \to \infty$ or $a_n \to -\infty$, then all its subsequences diverge in the same way. When a_n does not converge or diverge to $\pm\infty$, the situation is a bit more difficult because some subsequences may converge and others may diverge.

Example 3.14. Let $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ and $\alpha \in \mathbb{R}$. Since every interval contains an infinite number of rational numbers, it is possible to choose $\sigma(1) = \min\{k : |q_k - \alpha| < 1\}$. In general, assuming $\sigma(n)$ has been chosen, choose $\sigma(n+1) = \min\{k > \sigma(n) : |q_k - \alpha| < 1/n\}$. Such a choice is always possible because $\mathbb{Q} \cap (\alpha - 1/n, \alpha + 1/n) \setminus \{q_k : k \leq \sigma(n)\}$ is infinite. This induction yields a subsequence $q_{\sigma(n)}$ of q_n converging to α.

If a_n is a sequence and b_n is a convergent subsequence of a_n with $b_n \to L$, then L is called an *accumulation point* of a_n. A convergent sequence has only one accumulation point, but a divergent sequence may have many accumulation points. As seen in Example 3.14, a sequence may have all of $\mathbb{R}$ as its set of accumulation points.

To make some sense out of this, suppose a_n is a bounded sequence, and $T_n = \{a_k : k \geq n\}$. Define

$$\ell_n = \text{glb}\, T_n \text{ and } \mu_n = \text{lub}\, T_n.$$

[1]This theorem is named after Karl Weierstrass (1815–1897), a German mathematician, and Bernard Bolzano (1781–1848), a Bohemian mathematician. It was first proved in 1817 by Bolzano and was rediscovered by Weierstrass in about 1870.

Because $T_n \supset T_{n+1}$, it follows that for all $n \in \mathbb{N}$,

$$\ell_1 \le \ell_n \le \ell_{n+1} \le \mu_{n+1} \le \mu_n \le \mu_1. \tag{3.3}$$

This shows ℓ_n is an increasing sequence bounded above by μ_1 and μ_n is a decreasing sequence bounded below by ℓ_1. Theorem 3.11 implies both ℓ_n and μ_n converge. If $\ell_n \to \ell$ and $\mu_n \to \mu$, (3.3) shows for all n,

$$\ell_n \le \ell \le \mu \le \mu_n. \tag{3.4}$$

Suppose $b_n \to \beta$ is any convergent subsequence of a_n. From the definitions of ℓ_n and μ_n, it is seen that $\ell_n \le b_n \le \mu_n$ for all n. Now (3.4) shows $\ell \le \beta \le \mu$.

The normal terminology for ℓ and μ is given by the following definition.

Definition 3.17. Let a_n be a sequence. If a_n is bounded below, then the lower limit of a_n is

$$\liminf a_n = \lim_{n\to\infty} \operatorname{glb}\{a_k : k \ge n\}.$$

If a_n is bounded above, then the upper limit of a_n is

$$\limsup a_n = \lim_{n\to\infty} \operatorname{lub}\{a_k : k \ge n\}.$$

When a_n is unbounded, the lower and upper limits are set to appropriate infinite values, while recalling the familiar warnings about ∞ not being a number.

Example 3.15. Define

$$a_n = \begin{cases} 2 + 1/n, & n \text{ odd} \\ 1 - 1/n, & n \text{ even} \end{cases}.$$

Then

$$\mu_n = \operatorname{lub}\{a_k : k \ge n\} = \begin{cases} 2 + 1/n, & n \text{ odd} \\ 2 + 1/(n+1), & n \text{ even} \end{cases} \downarrow 2$$

and

$$\ell_n = \operatorname{glb}\{a_k : k \ge n\} = \begin{cases} 1 - 1/n, & n \text{ even} \\ 1 - 1/(n+1), & n \text{ odd} \end{cases} \uparrow 1.$$

So,

$$\limsup a_n = 2 > 1 = \liminf a_n.$$

Suppose a_n is bounded above with T_n, μ_n and μ as in the discussion preceding the definition. We claim there is a subsequence of a_n converging to $\limsup a_n$. The subsequence will be selected by induction.

Choose $\sigma(1) \in \mathbb{N}$ such that $a_{\sigma(1)} > \mu_1 - 1$.

Suppose $\sigma(n)$ has been selected for some $n \in \mathbb{N}$. Since $\mu_{\sigma(n)+1} = \operatorname{lub} T_{\sigma(n)+1}$, there must be an $a_m \in T_{\sigma(n)+1}$ such that $a_m > \mu_{\sigma(n)+1} - 1/(n+1)$. Then $m > \sigma(n)$ and we set $\sigma(n+1) = m$.

This inductively defines a subsequence $a_{\sigma(n)}$, where

$$\mu_{\sigma(n)} \geq \mu_{\sigma(n)+1} \geq a_{\sigma(n)} > \mu_{\sigma(n)+1} - \frac{1}{n+1}. \tag{3.5}$$

for all n. The left and right sides of (3.5) both converge to $\limsup a_n$, so the Squeeze Theorem implies $a_{\sigma(n)} \to \limsup a_n$.

In the cases when $\limsup a_n = \infty$ and $\limsup a_n = -\infty$, it is left to the reader to show there is a subsequence $b_n \to \limsup a_n$.

Similar arguments can be made for $\liminf a_n$. Assuming $\sigma(n)$ has been chosen for some $n \in \mathbb{N}$, To summarize: If β is an accumulation point of a_n, then

$$\liminf a_n \leq \beta \leq \limsup a_n.$$

In case a_n is bounded, both $\liminf a_n$ and $\limsup a_n$ are accumulation points of a_n and a_n converges iff $\liminf a_n = \lim_{n\to\infty} a_n = \limsup a_n$.

The following theorem has been proved.

Theorem 3.18. *Let a_n be a sequence.*

(a) *There are subsequences of a_n converging to* $\liminf a_n$ *and* $\limsup a_n$.

(b) *If α is an accumulation point of a_n, then* $\liminf a_n \leq \alpha \leq \limsup a_n$.

(c) $\liminf a_n = \limsup a_n \in \mathbb{R}$ *iff a_n converges. In this case, their common value is the limit of a_n.*

3.5 Cauchy Sequences

Often the biggest problem with showing that a sequence converges using the techniques we have seen so far is we must know ahead of time to what it converges. This is the “chicken and egg” problem mentioned above. An escape from this dilemma is provided by Cauchy sequences.

Definition 3.19. A sequence a_n is a *Cauchy sequence* [2] if for all $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|a_n - a_m| < \varepsilon$.

This definition is a bit more subtle than it might at first appear. It sort of says that all the terms of the sequence are close together from some point onward. The emphasis is on *all* the terms from some point onward. To stress this, first consider a negative example.

Example 3.16. Suppose $a_n = \sum_{k=1}^{n} 1/k$ for $n \in \mathbb{N}$. There's a trick for showing the sequence a_n diverges. First, note that a_n is strictly increasing. For any $n \in \mathbb{N}$, consider

$$\begin{aligned} a_{2^n-1} &= \sum_{k=1}^{2^n-1} \frac{1}{k} = \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \frac{1}{2^j+k} \\ &> \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \frac{1}{2^{j+1}} = \sum_{j=0}^{n-1} \frac{1}{2} = \frac{n}{2} \to \infty \end{aligned}$$

Hence, the subsequence a_{2^n-1} is unbounded and Theorem 3.14 shows the sequence a_n diverges. (To see how this works, write out the first few sums of the form a_{2^n-1}.)

On the other hand, $|a_{n+1} - a_n| = 1/(n+1) \to 0$ and indeed, if m is fixed, $|a_{n+m} - a_n| \to 0$. This makes it seem as though the terms are getting close together, as in the definition of a Cauchy sequence. But, a_n is not a Cauchy sequence, as shown by the following theorem.

Theorem 3.20. *A sequence converges iff it is a Cauchy sequence.*

Proof. ($\Rightarrow$) Suppose $a_n \to L$ and $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \varepsilon/2$. If $m, n \geq N$, then

$$|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_m| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows a_n is a Cauchy sequence.

($\Leftarrow$) Let a_n be a Cauchy sequence. First, we claim that a_n is bounded. To see this, let $\varepsilon = 1$ and choose $N \in \mathbb{N}$ such that $n, m \geq N$ implies $|a_n - a_m| < 1$. In this case, $a_N - 1 < a_n < a_N + 1$ for all $n \geq N$, so $\{a_n : n \geq N\}$ is a bounded set. The set finite set $\{a_n : n < N\}$ is also bounded. Since $\{a_n : n \in \mathbb{N}\}$ is the union of these two bounded sets, it too must be bounded.

Because a_n is a bounded sequence, Theorem 3.16 implies it has a convergent subsequence $b_n = a_{\sigma(n)} \to L$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ so that $n, m \geq N$

[2] Augustin-Louis Cauchy, 1789–1857.

implies $|a_n - a_m| < \varepsilon/2$ and $|b_n - L| < \varepsilon/2$. If $n \geq N$, then $\sigma(n) \geq n \geq N$ and

$$\begin{aligned} |a_n - L| &= |a_n - b_n + b_n - L| \\ &\leq |a_n - b_n| + |b_n - L| \\ &= |a_n - a_{\sigma(n)}| + |b_n - L| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Therefore, $a_n \to L$. □

The fact that Cauchy sequences converge is yet another equivalent version of completeness. In fact, most advanced texts define completeness as "Cauchy sequences converge." This is convenient in general spaces because the definition of a Cauchy sequence only needs the metric on the space and none of its other structure.

A typical example of the usefulness of Cauchy sequences is given below.

Definition 3.21. A sequence x_n is *contractive* if there is a $c \in (0,1)$ such that $|x_{k+1} - x_k| \leq c|x_k - x_{k-1}|$ for all $k > 1$. c is called the *contraction constant*.

Theorem 3.22. *If a sequence is contractive, then it converges.*

Proof. Let x_k be a contractive sequence with contraction constant $c \in (0,1)$.

We first claim that if $n \in \mathbb{N}$, then

$$|x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2|. \tag{3.6}$$

This is proved by induction. When $n = 1$, the statement is

$$|x_1 - x_2| \leq c^0|x_1 - x_2| = |x_1 - x_2|,$$

which is trivially true. Suppose that $|x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2|$ for some $n \in \mathbb{N}$. Then, from the definition of a contractive sequence and the induction hypothesis,

$$|x_{n+1} - x_{n+2}| \leq c|x_n - x_{n+1}| \leq c\left(c^{n-1}|x_1 - x_2|\right) = c^n|x_1 - x_2|.$$

This shows the claim is true in the case $n + 1$. Therefore, by induction, the claim is true for all $n \in \mathbb{N}$.

To show x_n is a Cauchy sequence, let $\varepsilon > 0$. Since $c^n \to 0$, we can choose $N \in \mathbb{N}$ so that

$$\frac{c^{N-1}}{(1-c)}|x_1 - x_2| < \varepsilon. \tag{3.7}$$

Let $n > m \geq N$. Then

$$\begin{aligned}|x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots - x_{m+1} + x_{m+1} - x_m| \\ &\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|\end{aligned}$$

Now, use (3.6) on each of these terms.

$$\begin{aligned}&\leq c^{n-2}|x_1 - x_2| + c^{n-3}|x_1 - x_2| + \cdots + c^{m-1}|x_1 - x_2| \\ &= |x_1 - x_2|(c^{n-2} + c^{n-3} + \cdots + c^{m-1})\end{aligned}$$

Apply the formula for a geometric sum.

$$\begin{aligned}&= |x_1 - x_2| c^{m-1} \frac{1 - c^{n-m}}{1 - c} \\ &< |x_1 - x_2| \frac{c^{m-1}}{1 - c} \qquad (3.8)\end{aligned}$$

Use (3.7) to estimate the following.

$$\begin{aligned}&\leq |x_1 - x_2| \frac{c^{N-1}}{1 - c} \\ &< |x_1 - x_2| \frac{\varepsilon}{|x_1 - x_2|} \\ &= \varepsilon\end{aligned}$$

This shows x_n is a Cauchy sequence and must converge by Theorem 3.20. □

Example 3.17. Let $-1 < r < 1$ and define the sequence $s_n = \sum_{k=0}^{n} r^k$. (You no doubt recognize this as the sequence of partial sums *geometric series* from your calculus course.) If $r = 0$, the convergence of s_n is trivial. So, suppose $r \neq 0$. In this case,

$$\frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = \left| \frac{r^{n+1}}{r^n} \right| = |r| < 1$$

and s_n is contractive. Theorem 3.22 implies s_n converges.

Example 3.18. Suppose $f(x) = 2 + 1/x$, $a_1 = 2$ and $a_{n+1} = f(a_n)$ for $n \in \mathbb{N}$. It is evident that $a_n \geq 2$ for all n. Some algebra gives

$$\left| \frac{a_{n+1} - a_n}{a_n - a_{n-1}} \right| = \left| \frac{f(f(a_{n-1})) - f(a_{n-1})}{f(a_{n-1}) - a_{n-1}} \right| = \frac{1}{1 + 2a_{n-1}} \leq \frac{1}{5}.$$

This shows a_n is a contractive sequence and, according to Theorem 3.22, $a_n \to L$ for some $L \geq 2$. Since, $a_{n+1} = 2 + 1/a_n$, taking the limit as $n \to \infty$ of both sides gives $L = 2 + 1/L$. A bit more algebra shows $L = 1 + \sqrt{2}$.

L is called a *fixed point* of the function f; i.e. $f(L) = L$. Many approximation techniques for solving equations involve such iterative techniques depending upon contraction to find fixed points.

The calculations in the proof of Theorem 3.22 give the means to approximate the fixed point to within an allowable error. Looking at line (3.8), notice

$$|x_n - x_m| < |x_1 - x_2| \frac{c^{m-1}}{1-c}.$$

Let $n \to \infty$ in this inequality to arrive at the error estimate

$$|L - x_m| \leq |x_1 - x_2| \frac{c^{m-1}}{1-c}. \tag{3.9}$$

In Example 3.18, $a_1 = 2$, $a_2 = 5/2$ and $c \leq 1/5$. Suppose we want to approximate L to 5 decimal places of accuracy. It suffices to find n satisfying $|a_n - L| < 5 \times 10^{-6}$. Using (3.9), with $m = 9$ shows

$$|a_1 - a_2| \frac{c^{m-1}}{1-c} \leq 1.6 \times 10^{-6}.$$

Some arithmetic gives $a_9 \approx 2.41421$. The calculator value of

$$L = 1 + \sqrt{2} \approx 2.414213562,$$

confirming our estimate.

3.6 Exercises

Exercise 3.1. Let the sequence $a_n = \dfrac{6n-1}{3n+2}$. Use the definition of convergence for a sequence to show a_n converges.

Exercise 3.2. If a_n is a sequence such that $a_{2n} \to L$ and $a_{2n+1} \to L$, then $a_n \to L$.

Exercise 3.3. Let a_n be a sequence such that $a_{2n} \to A$ and $a_{2n} - a_{2n-1} \to 0$. Then $a_n \to A$.

Exercise 3.4. If a_n is a sequence of positive numbers converging to 0, then $\sqrt{a_n} \to 0$.

Exercise 3.5. Find examples of sequences a_n and b_n such that $a_n \to 0$ and $b_n \to \infty$ such that

(a) $a_n b_n \to 0$

(b) $a_n b_n \to \infty$

(c) $\lim_{n\to\infty} a_n b_n$ does not exist, but $a_n b_n$ is bounded.

(d) Given $c \in \mathbb{R}$, $a_n b_n \to c$.

Exercise 3.6. If x_n and y_n are sequences such that $\lim_{n\to\infty} x_n = L \neq 0$ and $\lim_{n\to\infty} x_n y_n$ exists, then $\lim_{n\to\infty} y_n$ exists.

Exercise 3.7. Determine the limit of $a_n = \sqrt[n]{n!}$. (Hint: If n is even, then show $n! > (n/2)^{n/2}$.)

Exercise 3.8. If $\sigma : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then $\sigma(n) \geq n$ for all $n \in \mathbb{N}$.

Exercise 3.9. Let $a_n = \sum_{k=n+1}^{2n} 1/k$.

(a) Prove a_n is a Cauchy sequence.

(b) Determine $\lim_{n\to\infty} a_n$

Exercise 3.10. Every unbounded sequence contains a monotonic subsequence.

Exercise 3.11. Find a sequence a_n such that given $x \in [0,1]$, there is a subsequence b_n of a_n such that $b_n \to x$.

Exercise 3.12. A sequence a_n converges to 0 iff $|a_n|$ converges to 0.

Exercise 3.13. Define the sequence $a_n = \sqrt{n}$ for $n \in \mathbb{N}$. Show that $|a_{n+1} - a_n| \to 0$, but a_n is not a Cauchy sequence.

Exercise 3.14. Let f_n be the Fibonacci sequence, as defined in Example 3.6, and $\phi = (1+\sqrt{5})/2$. Prove

$$f_n = \frac{\phi^n - (-\phi)^{-n}}{\sqrt{5}}. \tag{3.11}$$

(The number $\phi \approx 1.61803$ is known as the *golden ratio* and the expression given in (3.11) is known as *Binet's formula*.)

Exercise 3.15. Let $a_1, \dots, a_n$ and $b_1, \dots, b_n$ be real numbers. Then

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 \le \left(\sum_{k=1}^{n} a_k^2\right)\left(\sum_{k=1}^{n} b_k^2\right). \tag{3.12}$$

(Hint: Let $f(x) = (a_1 x - b_1)^2 + \cdots + (a_n x - b_n)^2$.) This is called the Cauchy-Schwarz inequality.

Exercise 3.16. Suppose a sequence is defined by $a_1 = 0$, $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for $n \ge 2$. Prove a_n converges, and determine its limit.

Exercise 3.17. If the sequence a_n is defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 1}$, then show a_n converges and determine its limit.

Exercise 3.18. Let $a_1 = 3$ and $a_{n+1} = 2 - 1/a_n$ for $n \in \mathbb{N}$. Analyze the sequence.

Exercise 3.19. If a_n is a sequence such that $\lim_{n\to\infty} |a_{n+1}/a_n| = \rho < 1$, then $a_n \to 0$.

Exercise 3.20. If $a > 0$, then $\lim a^{1/n} = 1$.

The next four exercises outline a proof of the following theorem, as well as an example of its use.

Theorem 3.23. *If a_n is a sequence such that*

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \rho, \tag{3.14}$$

where ρ can be infinite, then $|a_n|^{1/n} \to \rho$. (This is sometimes called Cauchy's second limit theorem.)

Exercise 3.21. Prove Theorem 3.23 when $\rho = 0$.

Exercise 3.22. Prove Theorem 3.23 when $\rho \in (0, \infty)$.

Exercise 3.23. Prove Theorem 3.23 when $\rho = \infty$.

Exercise 3.24. Find

$$\lim_{n\to\infty}\left(\frac{2}{1}\left(\frac{3}{2}\right)^2\left(\frac{4}{3}\right)^3\cdots\left(\frac{n+1}{n}\right)^n\right)^{1/n}.$$

Exercise 3.25. Let a_n and b_n be sequences. Prove that both sequences a_n and b_n converge iff both $a_n + b_n$ and $a_n - b_n$ converge.

Exercise 3.26. Let a_n be a bounded sequence. Prove that given any $\varepsilon > 0$, there is an interval I with length ε such that $\{n : a_n \in I\}$ is infinite. Is it necessary that a_n be bounded?

Exercise 3.27. A sequence a_n *converges in the mean* if $\bar{a}_n = \frac{1}{n}\sum_{k=1}^{n} a_k$ converges. Prove that if $a_n \to L$, then $\bar{a}_n \to L$, but the converse is not true. (This is sometimes called Cauchy's first limit theorem.)

Exercise 3.28. Find a sequence x_n such that for all $n \in \mathbb{N}$ there is a subsequence of x_n converging to n.

Exercise 3.29. If a_n is a Cauchy sequence whose terms are integers, what can you say about the sequence?

Exercise 3.30. Show $a_n = \sum_{k=0}^{n} 1/k!$ is a Cauchy sequence.

Exercise 3.31. If a_n is a sequence such that every subsequence of a_n has a further subsequence converging to L, then $a_n \to L$.

Exercise 3.32. If $a, b \in (0, \infty)$, then show $\sqrt[n]{a^n + b^n} \to \max\{a, b\}$.

Exercise 3.33. If $0 < \alpha < 1$ and s_n is a sequence satisfying $|s_{n+1}| < \alpha|s_n|$, then $s_n \to 0$.

Exercise 3.34. A sequence a_n converges if and only if given $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|a_{N+p} - a_N| < \varepsilon$ for all $p \in \mathbb{N}$.

Exercise 3.35. If $c \geq 1$ in the definition of a contractive sequence, can the sequence converge?

Exercise 3.36. If a_n is a convergent sequence and b_n is a sequence such that $|a_m - a_n| \geq |b_m - b_n|$ for all $m, n \in \mathbb{N}$, then b_n converges.

Exercise 3.37. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \to L$, then $\sqrt{a_n} \to \sqrt{L}$.

Exercise 3.38. If a_n is a Cauchy sequence and b_n is a subsequence of a_n such that $b_n \to L$, then $a_n \to L$.

Exercise 3.39. Let $x_1 = 3$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Analyze the sequence.

Exercise 3.40. Let a_n be a sequence. $a_n \to L$ iff $\limsup a_n = L = \liminf a_n$.

Exercise 3.41. Is $\limsup(a_n + b_n) = \limsup a_n + \limsup b_n$?

Exercise 3.42. If a_n is a sequence of positive numbers, then $1/\liminf a_n = \limsup 1/a_n$. (Interpret $1/\infty = 0$ and $1/0 = \infty$)

Exercise 3.43. $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$

Exercise 3.44. $a_n = 1/n$ is not contractive.

Exercise 3.45.

The equation $x^3 - 4x + 2 = 0$ has one real root lying between 0 and 1. Find a sequence of rational numbers converging to this root. Use this sequence to approximate the root to five decimal places.

Exercise 3.46. Define

$$a_n = \sqrt{1 + \sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots + \sqrt{1}}}}} \quad (n \text{ square roots}).$$

Show a_n converges and determine its limit.

Exercise 3.47. Prove or give a counterexample: If $a_n \to L$ and $\sigma : \mathbb{N} \to \mathbb{N}$ is bijective, then $b_n = a_{\sigma(n)}$ converges. Note that b_n might not be a subsequence of a_n. (b_n is called a *rearrangement* of a_n.)

Chapter 4

Series

Given a sequence a_n, in many contexts it is natural to ask about the sum of all the numbers in the sequence. If only a finite number of the a_n are nonzero, this is trivial—and not very interesting. If an infinite number of the terms aren't zero, the path becomes less obvious. Indeed, it's even somewhat questionable whether it makes sense at all to add an infinite list of numbers.

There are many approaches to this question. The method given below is the most common technique. Others are mentioned in the exercises.

4.1 What is a Series?

The idea behind adding up an infinite collection of numbers is a reduction to the well-understood idea of a sequence. This is a typical approach in mathematics: reduce a question to a previously solved problem.

Definition 4.1. Given a sequence a_n, the *series* having a_n as its *terms* is the new sequence

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The numbers s_n are called the *partial sums* of the series. If $s_n \to S \in \mathbb{R}$, then the series *converges to* S, which is called the *sum of the series.* This is normally written as

$$\sum_{k=1}^{\infty} a_k = S.$$

Otherwise, the series *diverges.*

The notation $\sum_{n=1}^{\infty} a_n$ is understood to stand for the sequence of partial sums of the series with terms a_n. When there is no ambiguity, this is often abbreviated to just $\sum a_n$. It is often convenient to use the notation

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots .$$

Example 4.1. If $a_n = (-1)^n$ for $n \in \mathbb{N}$, then $s_1 = -1$, $s_2 = -1 + 1 = 0$, $s_3 = -1 + 1 - 1 = -1$ and in general

$$s_n = \frac{(-1)^n - 1}{2}$$

does not converge because it oscillates between -1 and 0. Therefore, the series $\sum(-1)^n$ diverges.

Example 4.2 (Geometric Series). Recall that a sequence of the form $a_n = c\,r^{n-1}$ is called a geometric sequence. It gives rise to a series

$$\sum_{n=1}^{\infty} c\,r^{n-1} = c + cr + cr^2 + cr^3 + \cdots$$

called a *geometric series*. The number r is called the *ratio* of the series.

Suppose $a_n = r^{n-1}$ for $r \neq 1$. Then the partial sums are

$$s_1 = 1,\ s_2 = 1 + r,\ s_3 = 1 + r + r^2,\ \ldots$$

In general, it can be shown by induction (or even long division of polynomials) that

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} r^{k-1} = \frac{1 - r^n}{1 - r}. \tag{4.1}$$

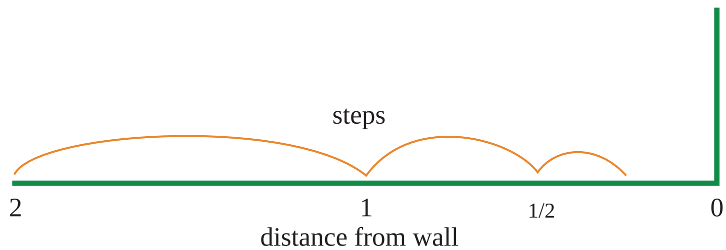
The convergence of s_n in (4.1) depends on the value of r. Letting $n \to \infty$, it's apparent that s_n diverges when $|r| > 1$ and converges to $1/(1-r)$ when $|r| < 1$. When $r = 1$, $s_n = n \to \infty$. When $r = -1$, it's essentially the same as Example 4.1, and therefore diverges. In summary,

$$\sum_{n=1}^{\infty} c\,r^{n-1} = \frac{c}{1-r}$$

for $|r| < 1$, and diverges when $|r| \geq 1$. This is called a *geometric series* with *ratio* r.

In some cases, the geometric series has an intuitively plausible limit. If you start two meters away from a wall and keep stepping halfway to the wall, no number of steps will get you to the wall, but a large number of steps will get you as close to the wall as you want. (See Figure 4.1.) So, the total distance stepped has limiting value 2. The total distance after n steps is the nth partial sum of a geometric series with ratio $r = 1/2$ and $c = 1$.

Figure 4.1: Stepping to the wall.



The geometric series is so important, it deserves its own theorem.

Theorem 4.2 (Geometric Series). *If $c, r \in \mathbb{R}$ with $c \neq 0$, then*

$$\sum_{n=1}^{\infty} cr^{n-1} = \frac{c}{1-r}$$

when $|r| < 1$ and diverges when $|r| \geq 1$.

Example 4.3 (Harmonic Series). The series $\sum_{n=1}^{\infty} 1/n$ is called the *harmonic* series. It was shown in Example 3.16 that the harmonic series diverges.

Example 4.4. The terms of the sequence

$$a_n = \frac{1}{n^2+n}, \quad n \in \mathbb{N}.$$

can be decomposed into partial fractions as

$$a_n = \frac{1}{n} - \frac{1}{n+1}.$$

If s_n is the series having a_n as its terms, then $s_1 = 1/2 = 1 - 1/2$. We claim that $s_n = 1 - 1/(n+1)$ for all $n \in \mathbb{N}$. To see this, suppose $s_k = 1 - 1/(k+1)$ for some $k \in \mathbb{N}$. Then

$$s_{k+1} = s_k + a_{k+1} = 1 - \frac{1}{k+1} + \left(\frac{1}{k+1} - \frac{1}{k+2}\right) = 1 - \frac{1}{k+2}$$

and the claim is established by induction. Now it's easy to see that

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} = \lim_{n\to\infty}\left(1 - \frac{1}{n+2}\right) = 1.$$

This is an example of a *telescoping* series. The name is apparently based on the idea that the middle terms of the series cancel, causing the series to collapse like a hand-held telescope.

The following theorem is an easy consequence of the properties of sequences shown in Theorem 3.8.

Theorem 4.3. *Let $\sum a_n$ and $\sum b_n$ be convergent series.*

(a) *If $c \in \mathbb{R}$, then $\sum c\, a_n = c \sum a_n$.*

(b) $\sum (a_n + b_n) = \sum a_n + \sum b_n$.

(c) $a_n \to 0$

Proof. Let $A_n = \sum_{k=1}^{n} a_k$ and $B_n = \sum_{k=1}^{n} b_k$ be the sequences of partial sums for each of the two series. By assumption, there are numbers A and B where $A_n \to A$ and $B_n \to B$.

(a) $\sum_{k=1}^{n} c\, a_k = c \sum_{k=1}^{n} a_k = cA_n \to cA$.

(b) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = A_n + B_n \to A + B$.

(c) For $n > 1$, $a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = A_n - A_{n-1} \to A - A = 0$. □

Notice that the first two parts of Theorem 4.3 show that the set of all convergent series is closed under linear combinations.

Theorem 4.3(c) is very useful because its contrapositive provides the most basic test for divergence.

Corollary 4.4 (Going to Zero Test)**.** *If $a_n \not\to 0$, then $\sum a_n$ diverges.*

Many students have made the mistake of reading too much into Corollary 4.4. It can only be used to show divergence. When the terms of a series do tend to zero, that does not guarantee convergence. Example 4.3, shows Theorem 4.3(c) is necessary, but not sufficient for convergence.

Another useful observation is that the partial sums of a convergent series are a Cauchy sequence. The Cauchy criterion for sequences can be rephrased for series as the following theorem, the proof of which is Exercise 4.5.

Theorem 4.5 (Cauchy Criterion for Series)**.** *Let $\sum a_n$ be a series. The following statements are equivalent.*

(a) *$\sum a_n$ converges.*

(b) *For every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $n \ge m \ge N$, then*

$$\left| \sum_{i=m}^{n} a_i \right| < \varepsilon.$$

4.2 Positive Series

Most of the time, it is very hard or impossible to determine the exact limit of a convergent series. We must satisfy ourselves with determining whether a series converges, and then approximating its sum. For this reason, the study of series usually involves learning a collection of theorems that might answer whether a given series converges, but don't tell us to what it converges. These theorems are usually called the *convergence tests*. The reader probably remembers a battery of such tests from her calculus course. There is a myriad of such tests, and the standard ones are presented in the next few sections, along with a few of those less widely used.

Since convergence of a series is determined by convergence of the sequence of its partial sums, the easiest series to study are those with well-behaved partial sums. Series with monotone sequences of partial sums are certainly the simplest such series.

Definition 4.6. The series $\sum a_n$ is a *positive series*, if $a_n \geq 0$ for all n.

The advantage of a positive series is that its sequence of partial sums is nonnegative and increasing. Since an increasing sequence converges if and only if it is bounded above, there is a simple criterion to determine whether a positive series converges: it converges if and only if its partial sums are a bounded sequence. All of the standard convergence tests for positive series exploit this criterion.

4.2.1 The Most Common Convergence Tests

All beginning calculus courses contain several simple tests to determine whether positive series converge. Most of them are presented below.

Comparison Tests

The most basic convergence tests are the comparison tests. In these tests, the behavior of one series is inferred from that of another series. Although they're easy to use, there is one often fatal catch: in order to use a comparison test, you must have a known series to which you can compare the mystery series. For this reason, a wise mathematician collects example series for her toolbox. The more samples in the toolbox, the more powerful are the comparison tests.

Theorem 4.7 (Comparison Test)**.** *Suppose $\sum a_n$ and $\sum b_n$ are positive series with $a_n \leq b_n$ for all n.*

(a) *If $\sum b_n$ converges, then so does $\sum a_n$.*

(b) *If $\sum a_n$ diverges, then so does $\sum b_n$.*

Proof. Let A_n and B_n be the partial sums of $\sum a_n$ and $\sum b_n$, respectively. It follows from the assumptions that A_n and B_n are increasing and for all $n \in \mathbb{N}$,

$$A_n \leq B_n. \tag{4.2}$$

If $\sum b_n = B$, then (4.2) implies B is an upper bound for A_n, and $\sum a_n$ converges.

On the other hand, if $\sum a_n$ diverges, $A_n \to \infty$ and the Sandwich Theorem 3.9(b) shows $B_n \to \infty$. □

Example 4.5. Example 4.3 shows that $\sum 1/n$ diverges. If $p \leq 1$, then $1/n^p \geq 1/n$, and Theorem 4.7 implies $\sum 1/n^p$ diverges.

Example 4.6. The series $\sum \sin^2 n/2^n$ converges because

$$\frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$$

for all n and the geometric series $\sum 1/2^n = 1$.

Theorem 4.8 (Cauchy's Condensation Test[1]). *Suppose a_n is a decreasing sequence of nonnegative numbers. Then*

$$\sum a_n \text{ converges iff } \sum 2^n a_{2^n} \text{ converges.}$$

Proof. Since a_n is decreasing, for $n \in \mathbb{N}$,

$$\sum_{k=2^n}^{2^{n+1}-1} a_k \leq 2^n a_{2^n} \leq 2 \sum_{k=2^{n-1}}^{2^n-1} a_k. \tag{4.3}$$

(See Figure 4.2.) Adding for $1 \leq n \leq m$ gives

$$\sum_{k=2}^{2^{m+1}-1} a_k \leq \sum_{k=1}^{m} 2^k a_{2^k} \leq 2 \sum_{k=1}^{2^m-1} a_k. \tag{4.4}$$

First suppose $\sum a_n$ converges to S. The right-hand inequality of (4.4) shows $\sum_{k=1}^{m} 2^k a_{2^k} < 2S$ and $\sum 2^k a_{2^k}$ must converge. On the other hand, if $\sum a_n$ diverges, then the left-hand side of (4.4) is unbounded, forcing $\sum 2^k a_{2^k}$ to diverge. □

[1] The series $\sum 2^n a_{2^n}$ is sometimes called the *condensed* series associated with $\sum a_n$.

$$a_1 + \underbrace{a_2 + a_3}_{\leq 2a_2} + \underbrace{a_4 + a_5 + a_6 + a_7}_{\leq 4a_4} + \underbrace{a_8 + a_9 + \cdots + a_{15}}_{\leq 8a_8} + a_{16} + \cdots$$

$$\underbrace{a_1}_{\geq a_2} + \underbrace{a_2 + a_3}_{\geq 2a_4} + \underbrace{a_4 + a_5 + a_6 + a_7}_{\geq 4a_8} + \underbrace{a_8 + a_9 + \cdots + a_{15}}_{\geq 8a_{16}} + a_{16} + \cdots$$

Figure 4.2: This diagram shows the groupings used in inequality (4.3).

Example 4.7 (p-series). For fixed $p \in \mathbb{R}$, the series $\sum 1/n^p$ is called a *p-series*. The special case when $p = 1$ is the *harmonic* series. Notice

$$\sum \frac{2^n}{(2^n)^p} = \sum \left(2^{1-p}\right)^n$$

is a geometric series with ratio 2^{1-p}, so it converges only when $2^{1-p} < 1$. Since $2^{1-p} < 1$ only when $p > 1$, it follows from the Cauchy Condensation Test that the p-series converges when $p > 1$ and diverges when $p \leq 1$. (Of course, the divergence half of this was already known from Example 4.5.)

The p-series are often useful for the Comparison Test, and also occur in many areas of advanced mathematics such as harmonic analysis and number theory.

Theorem 4.9 (Limit Comparison Test). *Suppose $\sum a_n$ and $\sum b_n$ are positive series with*

$$\alpha = \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} = \beta. \tag{4.5}$$

(a) *If $\alpha \in (0, \infty)$ and $\sum a_n$ converges, then so does $\sum b_n$, and if $\sum b_n$ diverges, then so does $\sum a_n$.*

(b) *If $\beta \in (0, \infty)$ and $\sum a_n$ diverges, then so does $\sum b_n$, and if $\sum b_n$ converges, then so does $\sum a_n$.*

Proof. To prove (a), suppose $\alpha > 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \implies \frac{\alpha}{2} < \frac{a_n}{b_n}. \tag{4.6}$$

If $n > N$, then (4.6) gives

$$\frac{\alpha}{2} \sum_{k=N}^{n} b_k < \sum_{k=N}^{n} a_k \tag{4.7}$$

If $\sum a_n$ converges, then (4.7) shows the partial sums of $\sum b_n$ are bounded and $\sum b_n$ converges. If $\sum b_n$ diverges, then (4.7) shows the partial sums of $\sum a_n$ are unbounded, and $\sum a_n$ must diverge.

The proof of (b) is similar. □

The following easy corollary is the form this test takes in most calculus books. It's easier to use than Theorem 4.9 and suffices most of the time.

Corollary 4.10 (Limit Comparison Test). *Suppose $\sum a_n$ and $\sum b_n$ are positive series with*

$$\alpha = \lim_{n\to\infty} \frac{a_n}{b_n}. \tag{4.8}$$

If $\alpha \in (0, \infty)$, then $\sum a_n$ and $\sum b_n$ simultaneously converge or diverge.

Example 4.8. To test the series $\sum \frac{1}{2^n - n}$ for convergence, let

$$a_n = \frac{1}{2^n - n} \text{ and } b_n = \frac{1}{2^n}.$$

Then

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1/(2^n - n)}{1/2^n} = \lim_{n\to\infty} \frac{2^n}{2^n - n} = \lim_{n\to\infty} \frac{1}{1 - n/2^n} = 1 \in (0, \infty).$$

Since $\sum 1/2^n = 1$, the original series converges by the Limit Comparison Test.

Geometric Series-Type Tests

The most important series is undoubtedly the geometric series. Several standard tests are basically comparisons to geometric series.

Theorem 4.11 (Root Test). *Suppose $\sum a_n$ is a positive series and*

$$\rho = \limsup a_n^{1/n}.$$

If $\rho < 1$, then $\sum a_n$ converges. If $\rho > 1$, then $\sum a_n$ diverges.

Proof. First, suppose $\rho < 1$ and $r \in (\rho, 1)$. There is an $N \in \mathbb{N}$ so that $a_n^{1/n} < r$ for all $n \geq N$. This is the same as $a_n < r^n$ for all $n \geq N$. Using this, it follows that when $n \geq N$,

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} a_k < \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} r^k < \sum_{k=1}^{N-1} a_k + \frac{r^N}{1-r}.$$

This shows the partial sums of $\sum a_n$ are bounded. Therefore, it must converge.

If $\rho > 1$, there is an increasing sequence of integers $k_n \to \infty$ such that $a_{k_n}^{1/k_n} > 1$ for all $n \in \mathbb{N}$. This shows $a_{k_n} > 1$ for all $n \in \mathbb{N}$. By Theorem 4.4, $\sum a_n$ diverges. $\square$

Example 4.9. For any $x \in \mathbb{R}$, the series $\sum |x^n|/n!$ converges. To see this, note that according to Exercise 3.7,

$$\left(\frac{|x^n|}{n!}\right)^{1/n} = \frac{|x|}{(n!)^{1/n}} \to 0 < 1.$$

Applying the Root Test shows the series converges.

Example 4.10. Consider the p-series $\sum 1/n$ and $\sum 1/n^2$. The first diverges and the second converges. Since $n^{1/n} \to 1$ and $n^{2/n} \to 1$, it can be seen that when $\rho = 1$, the Root Test in inconclusive.

Theorem 4.12 (Ratio Test)**.** *Suppose $\sum a_n$ is a positive series. Let*

$$r = \liminf \frac{a_{n+1}}{a_n} \leq \limsup \frac{a_{n+1}}{a_n} = R.$$

If $R < 1$, then $\sum a_n$ converges. If $r > 1$, then $\sum a_n$ diverges.

Proof. First, suppose $R < 1$ and $\rho \in (R, 1)$. There exists $N \in \mathbb{N}$ such that $a_{n+1}/a_n < \rho$ whenever $n \geq N$. This implies $a_{n+1} < \rho a_n$ whenever $n \geq N$. From this it's easy to prove by induction that $a_{N+m} < \rho^m a_N$ whenever $m \in \mathbb{N}$. It follows that, for $n > N$,

$$\begin{aligned}
\sum_{k=1}^{n} a_k &= \sum_{k=1}^{N} a_k + \sum_{k=N+1}^{n} a_k \\
&= \sum_{k=1}^{N} a_k + \sum_{k=1}^{n-N} a_{N+k} \\
&< \sum_{k=1}^{N} a_k + \sum_{k=1}^{n-N} a_N \rho^k \\
&< \sum_{k=1}^{N} a_k + \frac{a_N \rho}{1-\rho}.
\end{aligned}$$

Therefore, the partial sums of $\sum a_n$ are bounded, and $\sum a_n$ converges.

If $r > 1$, then choose $N \in \mathbb{N}$ so that $a_{n+1} > a_n$ for all $n \geq N$. It's now apparent that $a_n \not\to 0$. $\square$

In calculus books, the ratio test usually takes the following simpler form.

Corollary 4.13 (Ratio Test). *Suppose $\sum a_n$ is a positive series. Let*

$$r = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}.$$

If $r < 1$, then $\sum a_n$ converges. If $r > 1$, then $\sum a_n$ diverges.

From a practical viewpoint, the ratio test is often easier to apply than the root test. But, the root test is actually the stronger of the two in the sense that there are series for which the ratio test fails, but the root test succeeds. (See Exercise 12, for example.) This happens because

$$\liminf \frac{a_{n+1}}{a_n} \le \liminf a_n^{1/n} \le \limsup a_n^{1/n} \le \limsup \frac{a_{n+1}}{a_n}. \tag{4.9}$$

To see this, note the middle inequality is always true. To prove the right-hand inequality, choose $r > \limsup a_{n+1}/a_n$. It suffices to show $\limsup a_n^{1/n} \le r$. As in the proof of the ratio test, $a_{n+k} < r^k a_n$. This implies

$$a_{n+k} < r^{n+k}\frac{a_n}{r^n},$$

which leads to

$$a_{n+k}^{1/(n+k)} < r\left(\frac{a_n}{r^n}\right)^{1/(n+k)}.$$

Finally,

$$\limsup a_n^{1/n} = \limsup_{k\to\infty} a_{n+k}^{1/(n+k)} \le \limsup_{k\to\infty} r\left(\frac{a_n}{r^n}\right)^{1/(n+k)} = r.$$

The left-hand inequality is proved similarly.

The Integral Test

4.2.2 Kummer-Type Tests

This is an advanced section that can be omitted.

Most times the simple tests of the preceding section suffice. However, more difficult series require more delicate tests. There dozens of other, more specialized, convergence tests. Several of them are consequences of the following theorem.

Theorem 4.14 (Kummer's[2] Test). *Suppose $\sum a_n$ is a positive series, p_n is a sequence of positive numbers and*

$$\alpha = \liminf \left(p_n \frac{a_n}{a_{n+1}} - p_{n+1} \right) \le \limsup \left(p_n \frac{a_n}{a_{n+1}} - p_{n+1} \right) = \beta \tag{4.10}$$

If $\alpha > 0$, then $\sum a_n$ converges. If $\sum 1/p_n$ diverges and $\beta < 0$, then $\sum a_n$ diverges.

Proof. Let $s_n = \sum_{k=1}^n a_k$, suppose $\alpha > 0$ and choose $r \in (0, \alpha)$. There must be an $N > 1$ such that

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} > r, \ \forall n \ge N.$$

Rearranging this gives

$$p_n a_n - p_{n+1} a_{n+1} > r a_{n+1}, \ \forall n \ge N. \tag{4.11}$$

For $M > N$, (4.11) implies

$$\begin{aligned} \sum_{n=N}^{M} (p_n a_n - p_{n+1} a_{n+1}) &> \sum_{n=N}^{M} r a_{n+1} \\ p_N a_N - p_{M+1} a_{M+1} &> r(s_M - s_{N-1}) \\ p_N a_N - p_{M+1} a_{M+1} + r s_{N-1} &> r s_M \\ \frac{p_N a_N + r s_{N-1}}{r} &> s_M \end{aligned}$$

Since N is fixed, the left side is an upper bound for s_M, and it follows that $\sum a_n$ converges.

Next suppose $\sum 1/p_n$ diverges and $\beta < 0$. There must be an $N \in \mathbb{N}$ such that

$$p_n \frac{a_n}{a_{n+1}} - p_{n+1} < 0, \ \forall n \ge N.$$

This implies

$$p_n a_n < p_{n+1} a_{n+1}, \ \forall n \ge N.$$

Therefore, $p_n a_n > p_N a_N$ whenever $n > N$ and

$$a_n > p_N a_N \frac{1}{p_n}, \ \forall n \ge N.$$

Because N is fixed and $\sum 1/p_n$ diverges, the Comparison Test shows $\sum a_n$ diverges. □

[2] Ernst Kummer (1810–1893) was a German mathematician.

Kummer's test is powerful. In fact, it can be shown that, given any positive series, a judicious choice of the sequence p_n can always be made to determine whether it converges. (See Exercise 19, [21] and [20].) But, as stated, Kummer's test is not very useful because choosing p_n for a given series is often difficult. Experience has led to some standard choices that work with large classes of series. For example, Exercise 11 asks you to prove the choice $p_n = 1$ for all n reduces Kummer's test to the standard ratio test. Other useful choices are shown in the following theorems.

Theorem 4.15 (Raabe's[3] Test)**.** *Let $\sum a_n$ be a positive series such that $a_n > 0$ for all n. Define*

$$\alpha = \limsup_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) \geq \liminf_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \beta$$

If $\alpha > 1$, then $\sum a_n$ converges. If $\beta < 1$, then $\sum a_n$ diverges.

Proof. Let $p_n = n$ in Kummer's test, Theorem 4.14. □

When Raabe's test is inconclusive, there are even more delicate tests, such as the theorem given below.

Theorem 4.16 (Bertrand's[4] Test)**.** *Let $\sum a_n$ be a positive series such that $a_n > 0$ for all n. Define*

$$\begin{aligned}\alpha = \liminf_{n\to\infty} \ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right)\\ \leq \limsup_{n\to\infty} \ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) = \beta.\end{aligned} \tag{4.12}$$

If $\alpha > 1$, then $\sum a_n$ converges. If $\beta < 1$, then $\sum a_n$ diverges.

Proof. Let $p_n = n \ln n$ in Kummer's test. □

Example 4.11. Consider the series

$$\sum a_n = \sum \left(\prod_{k=1}^{n} \frac{2k}{2k+1}\right)^p. \tag{4.13}$$

It's of interest to know for what values of p it converges.

[3]Joseph Ludwig Raabe (1801–1859) was a Swiss mathematician.
[4]Joseph Bertrand (1822–1900) was a French mathematician.

An easy computation shows that $a_{n+1}/a_n \to 1$, so the ratio test is inconclusive.

Next, try Raabe's test. Manipulating

$$\lim_{n\to\infty} n\left(\left(\frac{a_n}{a_{n+1}}\right)^p - 1\right) = \lim_{n\to\infty} \frac{\left(\frac{2n+3}{2n+2}\right)^p - 1}{\frac{1}{n}}$$

it becomes a $0/0$ form and can be evaluated with L'Hospital's rule.[5]

$$\lim_{n\to\infty} \frac{n^2\left(\frac{3+2n}{2+2n}\right)^p p}{(1+n)\,(3+2n)} = \frac{p}{2}.$$

From Raabe's test, Theorem 4.15, it follows that the series converges when $p > 2$ and diverges when $p < 2$. Raabe's test is inconclusive when $p = 2$.

Now, suppose $p = 2$. Consider

$$\lim_{n\to\infty} \ln n\left(n\left(\frac{a_n}{a_{n+1}} - 1\right) - 1\right) = -\lim_{n\to\infty} \ln n \frac{(4+3n)}{4\,(1+n)^2} = 0$$

and Bertrand's test, Theorem 4.16, shows divergence.

The series (4.13) converges only when $p > 2$.

4.3 Absolute and Conditional Convergence

The tests given above are for the restricted case when a series has positive terms. If the stipulation that the series be positive is thrown out, things becomes considerably more complicated. But, as is often the case in mathematics, some problems can be attacked by reducing them to previously solved cases. The following definition and theorem show how to do this for some special cases.

Definition 4.17. Let $\sum a_n$ be a series. If $\sum |a_n|$ converges, then $\sum a_n$ is *absolutely convergent*. If it is convergent, but not absolutely convergent, then it is *conditionally convergent*.

Since $\sum |a_n|$ is a positive series, the preceding tests can be used to determine its convergence. The following theorem shows that this is also enough for convergence of the original series.

Theorem 4.18. *If $\sum a_n$ is absolutely convergent, then it is convergent.*

[5]See §7.5.2.

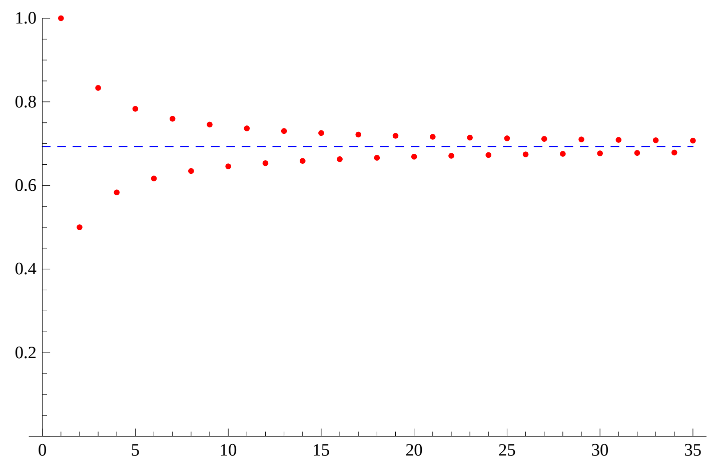


Figure 4.3: This plot shows the first 35 partial sums of the alternating harmonic series. It can be shown it converges to $\ln 2 \approx 0.6931$, which is the level of the dashed line. Notice how the odd partial sums decrease to $\ln 2$ and the even partial sums increase to $\ln 2$.

Proof. Let $\varepsilon > 0$. Theorem 4.5 yields an $N \in \mathbb{N}$ such that when $n \geq m \geq N$,

$$\varepsilon > \sum_{k=m}^{n} |a_k| \geq \left| \sum_{k=m}^{n} a_k \right| \geq 0.$$

Another application Theorem 4.5 finishes the proof. □

Example 4.12. The series $\sum (-1)^{n+1}/n$ is called the *alternating harmonic series*. (See Figure 4.3.) Since the harmonic series diverges, we see the alternating harmonic series is not absolutely convergent.

On the other hand, if $s_n = \sum_{k=1}^{n} (-1)^{k+1}/k$, then

$$s_{2n} = \sum_{k=1}^{n} \left(\frac{1}{2k-1} - \frac{1}{2k} \right) = \sum_{k=1}^{n} \frac{1}{2k(2k-1)}$$

is a positive series that converges by the Comparison Test. Since $|s_{2n} - s_{2n-1}| = 1/2n \to 0$, Exercise 17 shows s_{2n-1} must also converge to the same limit. Therefore, s_n converges and $\sum (-1)^{n+1}/n$ is conditionally convergent. It is shown in Exercise 23 that the alternating harmonic series converges to $\ln 2$.

To summarize: absolute convergence implies convergence, but convergence does not imply absolute convergence.

There are a few tests that address conditional convergence. Following are the most well-known.

Theorem 4.19 (Abel's[6] Test)**.** *Let a_n and b_n be sequences satisfying*

(a) $s_n = \sum_{k=1}^n a_k$ *is a bounded sequence.*

(b) $b_n \geq b_{n+1}$, $\forall n \in \mathbb{N}$

(c) $b_n \to 0$

Then $\sum a_n b_n$ converges.

To prove this theorem, the following lemma is needed.

Lemma 4.20 (Summation by Parts)**.** *For every pair of sequences a_n and b_n,*

$$\sum_{k=1}^n a_k b_k = b_{n+1} \sum_{k=1}^n a_k - \sum_{k=1}^n (b_{k+1} - b_k) \sum_{\ell=1}^k a_\ell$$

Proof. Let $s_0 = 0$ and $s_n = \sum_{k=1}^n a_k$ when $n \in \mathbb{N}$. Then

$$\begin{aligned}
\sum_{k=1}^n a_k b_k &= \sum_{k=1}^n (s_k - s_{k-1}) b_k \\
&= \sum_{k=1}^n s_k b_k - \sum_{k=1}^n s_{k-1} b_k \\
&= \sum_{k=1}^n s_k b_k - \left(\sum_{k=1}^n s_k b_{k+1} - s_n b_{n+1} \right) \\
&= b_{n+1} \sum_{k=1}^n a_k - \sum_{k=1}^n (b_{k+1} - b_k) \sum_{\ell=1}^k a_\ell
\end{aligned}$$

□

Proof. To prove the theorem, suppose $|\sum_{k=1}^n a_k| < M$ for all $n \in \mathbb{N}$. Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that $b_N < \varepsilon/2M$. If $N \leq m < n$, use Lemma 4.20 to write

$$\begin{aligned}
\left| \sum_{\ell=m}^n a_\ell b_\ell \right| &= \left| \sum_{\ell=1}^n a_\ell b_\ell - \sum_{\ell=1}^{m-1} a_\ell b_\ell \right| \\
&= \left| b_{n+1} \sum_{\ell=1}^n a_\ell - \sum_{\ell=1}^n (b_{\ell+1} - b_\ell) \sum_{k=1}^\ell a_k \right. \\
&\qquad \left. - \left(b_m \sum_{\ell=1}^{m-1} a_\ell - \sum_{\ell=1}^{m-1} (b_{\ell+1} - b_\ell) \sum_{k=1}^\ell a_k \right) \right|
\end{aligned}$$

[6] Niels Henrik Abel (1802–1829) was a Norwegian mathematician.

Using (a) gives

$$\leq (b_{n+1} + b_m)M + M \sum_{\ell=m}^{n} |b_{\ell+1} - b_\ell|$$

Now, use (b) to see

$$= (b_{n+1} + b_m)M + M \sum_{\ell=m}^{n} (b_\ell - b_{\ell+1})$$

and then telescope the sum to arrive at

$$\begin{aligned} &= (b_{n+1} + b_m)M + M(b_m - b_{n+1}) \\ &= 2Mb_m \\ &< 2M\frac{\varepsilon}{2M} \\ &= \varepsilon \end{aligned}$$

This shows $\sum_{\ell=1}^{n} a_\ell b_\ell$ satisfies Theorem 4.5, and therefore converges. $\square$

There's one special case of this theorem that's most often seen in calculus texts.

Corollary 4.21 (Alternating Series Test)**.** *If c_n decreases to 0, then the series $\sum(-1)^{n+1}c_n$ converges. Moreover, if $s_n = \sum_{k=1}^{n}(-1)^{k+1}c_k$ and $s_n \to s$, then $|s_n - s| < c_{n+1}$.*

Proof. Let $a_n = (-1)^{n+1}$ and $b_n = c_n$ in Theorem 4.19 to see the series converges to some number s. For $n \in \mathbb{N}$,

$$s_{2n} - s_{2n+2} = -c_{2n+1} + c_{2n+2} \leq 0 \text{ and } s_{2n+1} - s_{2n+3} = c_{2n+2} - c_{2n+3} \geq 0,$$

It must be that $s_{2n} \uparrow s$ and $s_{2n+1} \downarrow s$. For all $n \in \omega$,

$$0 \leq s_{2n+1} - s \leq s_{2n+1} - s_{2n+2} = c_{2n+2} \text{ and } 0 \leq s - s_{2n} \leq s_{2n+1} - s_{2n} = c_{2n+1}.$$

This shows $|s_n - s| < c_{n+1}$ for all n. $\square$

A series such as that in Corollary 4.21 is called an *alternating series.* More formally, if a_n is a sequence such that $a_n/a_{n+1} < 0$ for all n, then $\sum a_n$ is an alternating series. Informally, it just means the series alternates between positive and negative terms.

Example 4.13. Corollary 4.21 provides another way to prove the alternating harmonic series in Example 4.12 converges. Figures 4.3 and 4.4 show how the partial sums bounce up and down across the sum of the series.

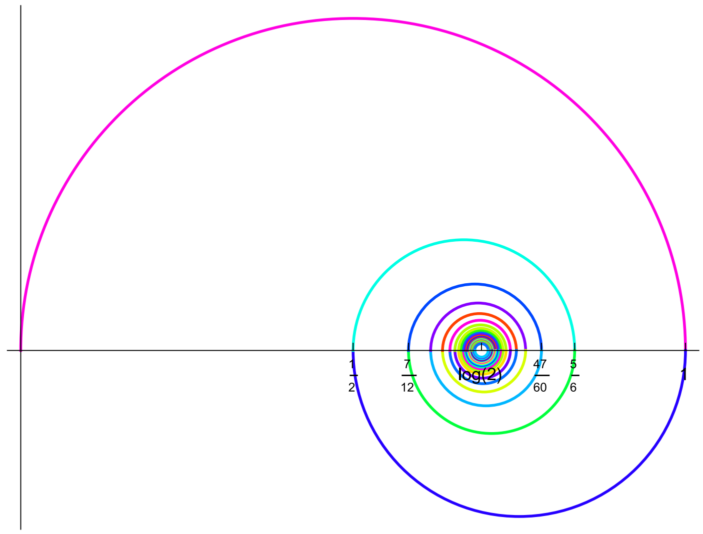


Figure 4.4: Here is a more whimsical way to visualize the partial sums of the alternating harmonic series.

4.4 Rearrangements of Series

This is an advanced section that can be omitted.

We want to use our standard intuition about adding lists of numbers when working with series. But, this intuition has been formed by working with finite sums and does not always work with series.

Example 4.14. Suppose $\sum(-1)^{n+1}/n = \gamma$ so that $\sum(-1)^{n+1}2/n = 2\gamma$. It's easy to show $\gamma > 1/2$. Consider the following calculation.

$$\begin{aligned}2\gamma &= \sum(-1)^{n+1}\frac{2}{n}\\ &= 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \cdots\end{aligned}$$

Rearrange and regroup.

$$\begin{aligned}&= (2-1) - \frac{1}{2} + \left(\frac{2}{3} - \frac{1}{3}\right) - \frac{1}{4} + \left(\frac{2}{5} - \frac{1}{5}\right) - \frac{1}{6} + \cdots\\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots\\ &= \gamma\end{aligned}$$

So, $\gamma = 2\gamma$ with $\gamma \neq 0$. Obviously, rearranging and regrouping of this series is a questionable thing to do.

In order to carefully consider the problem of rearranging a series, a precise definition is needed.

Definition 4.22. Let $\sigma : \mathbb{N} \to \mathbb{N}$ be a bijection and $\sum a_n$ be a series. The new series $\sum a_{\sigma(n)}$ is a rearrangement of the original series.

The problem with Example 4.14 is that the series is conditionally convergent. Such examples cannot happen with absolutely convergent series. For the most part, absolutely convergent series behave as we are intuitively led to expect.

Theorem 4.23. *If $\sum a_n$ is absolutely convergent and $\sum a_{\sigma(n)}$ is a rearrangement of $\sum a_n$, then $\sum a_{\sigma(n)} = \sum a_n$.*

Proof. Let $\sum a_n = s$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $N \leq m < n$ implies $\sum_{k=m}^{n} |a_k| < \varepsilon$. Choose $M \geq N$ such that

$$\{1, 2, \ldots, N\} \subset \{\sigma(1), \sigma(2), \ldots, \sigma(M)\}.$$

If $P > M$, then

$$\left| \sum_{k=1}^{P} a_k - \sum_{k=1}^{P} a_{\sigma(k)} \right| \leq \sum_{k=N+1}^{\infty} |a_k| \leq \varepsilon$$

and both series converge to the same number. $\square$

When a series is conditionally convergent, the result of a rearrangement is hard to predict. This is shown by the following surprising theorem.

Theorem 4.24 (Riemann[7] Rearrangement)**.** *If $\sum a_n$ is conditionally convergent and $c \in \mathbb{R} \cup \{-\infty, \infty\}$, then there is a rearrangement σ such that $\sum a_{\sigma(n)} = c$.*

To prove this, the following lemma is needed.

Lemma 4.25. *If $\sum a_n$ is conditionally convergent and*

$$b_n = \begin{cases} a_n, & a_n > 0 \\ 0, & a_n \leq 0 \end{cases} \quad \text{and} \quad c_n = \begin{cases} -a_n, & a_n < 0 \\ 0, & a_n \geq 0 \end{cases},$$

then both $\sum b_n$ and $\sum c_n$ diverge.

Proof. Suppose $\sum b_n$ converges. By assumption, $\sum a_n$ converges, so Theorem 4.3 implies

$$\sum c_n = \sum b_n - \sum a_n$$

[7]Bernhard Riemann (1826–1866) was a German mathematician.

converges. Another application of Theorem 4.3 shows

$$\sum |a_n| = \sum b_n + \sum c_n$$

converges. This is a contradiction of the assumption that $\sum a_n$ is conditionally convergent, so $\sum b_n$ cannot converge.

A similar contradiction arises under the assumption that $\sum c_n$ converges. $\square$

Proof. (Theorem 4.24) Let b_n and c_n be as in Lemma 4.25 and define the subsequence a_n^+ of b_n by removing those terms for which $b_n = 0$ and $a_n \neq 0$. Define the subsequence a_n^- of c_n by removing those terms for which $c_n = 0$. The series $\sum_{n=1}^\infty a_n^+$ and $\sum_{n=1}^\infty a_n^-$ are still divergent because only terms equal to zero have been removed from b_n and c_n.

Now, let $c \in \mathbb{R}$ and $m_0 = n_0 = 0$. According to Lemma 4.25, we can define the natural numbers

$$m_1 = \min\{n : \sum_{k=1}^n a_k^+ > c\} \text{ and } n_1 = \min\{n : \sum_{k=1}^{m_1} a_k^+ + \sum_{\ell=1}^n a_\ell^- < c\}.$$

If m_p and n_p have been chosen for some $p \in \mathbb{N}$, then define

$$m_{p+1} = \min\left\{ n : \sum_{k=0}^p \left(\sum_{\ell=m_k+1}^{m_{k+1}} a_\ell^+ - \sum_{\ell=n_k+1}^{n_{k+1}} a_\ell^- \right) + \sum_{\ell=m_p+1}^n a_\ell^+ > c \right\}$$

and

$$n_{p+1} = \min\left\{ n : \sum_{k=0}^p \left(\sum_{\ell=m_k+1}^{m_{k+1}} a_\ell^+ - \sum_{\ell=n_k+1}^{n_{k+1}} a_\ell^- \right) + \sum_{\ell=m_p+1}^{n_{p+1}} a_\ell^+ - \sum_{\ell=n_p+1}^n a_\ell^- < c \right\}.$$

Consider the series

$$\begin{aligned} a_1^+ + a_2^+ + \cdots + a_{m_1}^+ - a_1^- - a_2^- - \cdots - a_{n_1}^- \\ + a_{m_1+1}^+ + a_{m_1+2}^+ + \cdots + a_{m_2}^+ - a_{n_1+1}^- - a_{n_1+2}^- - \cdots - a_{n_2}^- \\ + a_{m_2+1}^+ + a_{m_2+2}^+ + \cdots + a_{m_3}^+ - a_{n_2+1}^- - a_{n_2+2}^- - \cdots - a_{n_3}^- \\ + \cdots \end{aligned} \tag{4.14}$$

It is clear this series is a rearrangement of $\sum_{n=1}^\infty a_n$ and the way in which m_p and n_p were chosen guarantee that

$$0 < \sum_{k=0}^{p-1}\left(\sum_{\ell=m_k+1}^{m_{k+1}} a_\ell^+ - \sum_{\ell=n_k+1}^{n_k} a_\ell^- + \sum_{k=m_p+1}^{m_p} a_k^+\right) - c \le a_{m_p}^+$$

and

$$0 < c - \sum_{k=0}^{p}\left(\sum_{\ell=m_k+1}^{m_{k+1}} a_\ell^+ - \sum_{\ell=n_k+1}^{n_k} a_\ell^-\right) \le a_{n_p}^-$$

Since both $a_{m_p}^+ \to 0$ and $a_{n_p}^- \to 0$, the result follows from the Squeeze Theorem.

The argument when c is infinite is left as Exercise 32. □

A moral to take from all this is that absolutely convergent series are robust and conditionally convergent series are fragile. Absolutely convergent series can be sliced and diced and mixed with careless abandon without getting surprising results. If conditionally convergent series are not handled with care, the results can be quite unexpected.

4.5 Exercises

Exercise 4.1. Prove Theorem 4.5.

Exercise 4.2. Analyze $\sum\left(1+\frac{1}{n}\right)$.

Exercise 4.3. If $\sum_{n=1}^\infty a_n$ is a convergent positive series, then does $\sum_{n=1}^\infty \frac{1}{1+a_n}$ converge?

Exercise 4.4. The series $\displaystyle\sum_{n=1}^\infty (a_n - a_{n+1})$ converges iff the sequence a_n converges.

Exercise 4.5. Prove or give a counter example: If $\sum |a_n|$ converges, then $na_n \to 0$.

Exercise 4.6. If the series $a_1 + a_2 + a_3 + \cdots$ converges to S, then so does

$$a_1 + 0 + a_2 + 0 + 0 + a_3 + 0 + 0 + 0 + a_4 + \cdots. \tag{4.15}$$

Exercise 4.7. Consider the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

Show that the ratio test is inconclusive with this series, but the root test is conclusive. What is its sum?

Exercise 4.8. If $\sum_{n=1}^{\infty} a_n$ converges and b_n is a bounded monotonic sequence, then $\sum_{n=1}^{\infty} a_n b_n$ converges.

Exercise 4.9. Let x_n be a sequence with range $\{0,1,2,3,4,5,6,7,8,9\}$. Prove that $\sum_{n=1}^{\infty} x_n 10^{-n}$ converges and its sum is in the interval $[0,1]$.

Exercise 4.10. Write $6.17272727272\cdots$ as a fraction.

Exercise 4.11. Prove the ratio test by setting $p_n = 1$ for all n in Kummer's test.

Exercise 4.12. Consider the series

$$1 + 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots = 4.$$

Show that the ratio test is inconclusive for this series, but the root test gives a positive answer.

Exercise 4.13. For what values of p does $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(n+1)\,(\ln(n+2))^p}$ converge?

Exercise 4.14. Does

$$\frac{1}{3} + \frac{1\times 2}{3\times 5} + \frac{1\times 2\times 3}{3\times 5\times 7} + \frac{1\times 2\times 3\times 4}{3\times 5\times 7\times 9} + \cdots$$

converge?

Exercise 4.15. For what values of p does

$$\left(\frac{1}{2}\right)^p + \left(\frac{1\cdot 3}{2\cdot 4}\right)^p + \left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)^p + \cdots$$

converge?

Exercise 4.16. Find sequences a_n and b_n satisfying:

(a) $a_n > 0, \forall n \in \mathbb{N}$ and $a_n \to 0$;

(b) $B_n = \sum_{k=1}^{n} b_k$ is a bounded sequence; and,

(c) $\sum_{n=1}^{\infty} a_n b_n$ diverges.

Exercise 4.17. Let a_n be a sequence such that $a_{2n} \to A$ and $a_{2n} - a_{2n-1} \to 0$. Then $a_n \to A$.

Exercise 4.18. Prove Bertrand's test, Theorem 4.16.

Exercise 4.19. Let $\sum a_n$ be a positive series. Prove that $\sum a_n$ converges if and only if there is a sequence of positive numbers p_n and $\alpha > 0$ such that

$$\lim_{n\to\infty} p_n \frac{a_n}{a_{n+1}} - p_{n+1} = \alpha.$$

(Hint: If $s = \sum a_n$ and $s_n = \sum_{k=1}^{n} a_k$, then let $p_n = (s - s_n)/a_n$.)

Exercise 4.20. Prove that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all $x \in \mathbb{R}$.

Exercise 4.21. Find all values of x for which $\sum_{k=0}^{\infty} k^2 (x+3)^k$ converges.

Exercise 4.22. For what values of x does the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1} \tag{4.21}$$

converge?

Exercise 4.23. Analyze $\sum \dfrac{\ln(1+1/n)}{n}$.

Exercise 4.24. For what values of x does $\sum_{n=1}^{\infty} \dfrac{(x+3)^n}{n4^n}$ converge absolutely, converge conditionally or diverge?

Exercise 4.25. For what values of x does $\sum_{n=1}^{\infty} \dfrac{n+6}{n^2(x-1)^n}$ converge absolutely, converge conditionally or diverge?

Exercise 4.26. For what positive values of α does $\sum_{n=1}^{\infty} \alpha^n n^{\alpha}$ converge?

Exercise 4.27. Prove that $\sum \cos \dfrac{n\pi}{3} \sin \dfrac{\pi}{n}$ converges.

Exercise 4.28. For a series $\sum_{k=1}^{\infty} a_n$ with partial sums s_n, define

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} s_n.$$

Prove that if $\sum_{k=1}^{\infty} a_n = s$, then $\sigma_n \to s$. Find an example where σ_n converges, but $\sum_{k=1}^{\infty} a_n$ does not. (If σ_n converges, the sequence is said to be *Cesàro summable*.)

Exercise 4.29. If a_n is a sequence with a subsequence b_n, then $\sum_{n=1}^{\infty} b_n$ is a *subseries* of $\sum_{n=1}^{\infty} a_n$. Prove that if every subseries of $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Exercise 4.30. If $\sum_{n=1}^{\infty} a_n$ is a convergent positive series, then so is $\sum_{n=1}^{\infty} a_n^2$. Give an example to show the converse is not true.

Exercise 4.31. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and there is a $p > 1$ such that $\lim_{n\to\infty} n^p a_n$ exists and is finite, then $\sum_{n=1}^{\infty} a_n$ converges. Is this true for $p = 1$?

Exercise 4.32. Finish the proof of Theorem 4.24.

Exercise 4.33. Leonhard Euler started with the equation

$$\frac{x}{x-1} + \frac{x}{1-x} = 0,$$

transformed it to

$$\frac{1}{1-1/x} + \frac{x}{1-x} = 0,$$

and then used geometric series to write it as

$$\cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + \cdots = 0. \tag{4.25}$$

Show how Euler did his calculation and find his mistake.

Exercise 4.34. Let $\sum a_n$ be a conditionally convergent series and $c \in \mathbb{R} \cup \{-\infty, \infty\}$. There is a sequence b_n such that $|b_n| = 1$ for all $n \in \mathbb{N}$ and $\sum a_n b_n = c$.

Chapter 5

The Topology of $\mathbb{R}$

This chapter introduces the basic ideas of point-set topology on $\mathbb{R}$. We'll see how a closer look the relation between limits and the standard metric reveal deeper properties of $\mathbb{R}$. These deeper properties are then used in subsequent chapters to prove some of the standard theorems of analysis.

5.1 Open and Closed Sets

Definition 5.1. A set $G \subset \mathbb{R}$ is *open* if for every $x \in G$ there is an $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \subset G$. A set $F \subset \mathbb{R}$ is *closed* if F^c is open.

The idea is that about every point of an open set, there is some room inside the set on both sides of the point. It is easy to see that any open interval (a, b) is an open set because if $a < x < b$ and $\varepsilon = \min\{x - a, b - x\}$, then $(x - \varepsilon, x + \varepsilon) \subset (a, b)$. It's obvious $\mathbb{R}$ itself is an open set.

On the other hand, any closed interval $[a, b]$ is a closed set. To see this, it must be shown its complement is open. Let $x \in [a, b]^c$ and $\varepsilon = \min\{|x - a|, |x - b|\}$. Then $(x - \varepsilon, x + \varepsilon) \cap [a, b] = \emptyset$, so $(x - \varepsilon, x + \varepsilon) \subset [a, b]^c$. Therefore, $[a, b]^c$ is open, and its complement, namely $[a, b]$, is closed.

A singleton set $\{a\}$ is closed. To see this, suppose $x \neq a$ and $\varepsilon = |x - a|$. Then $a \notin (x - \varepsilon, x + \varepsilon)$, and $\{a\}^c$ must be open. The definition of a closed set implies $\{a\}$ is closed.

Open and closed sets can get much more complicated than the intervals examined above. For example, similar arguments show $\mathbb{Z}$ is a closed set and $\mathbb{Z}^c$ is open. Both have an infinite number of disjoint pieces.

A common mistake is to assume all sets are either open or closed. Most sets are neither open nor closed. For example, if $S = [a, b)$ for some numbers

$a < b$, then no matter the size of $\varepsilon > 0$, neither $(a-\varepsilon, a+\varepsilon)$ nor $(b-\varepsilon, b+\varepsilon)$ are contained in S or S^c.

Theorem 5.2. *Following are the basic properties of open sets.*

(a) *Both $\emptyset$ and $\mathbb{R}$ are open.*

(b) *The union of any collection of open sets is an open set.*

(c) *The intersection of a finite collection of open sets is open.*

Proof. (a) $\emptyset$ is open vacuously. $\mathbb{R}$ is obviously open.

(b) If $x \in \bigcup_{\lambda\in\Lambda} G_\lambda$, then there is a $\lambda_x \in \Lambda$ such that $x \in G_{\lambda_x}$. Since G_{λ_x} is open, there is an $\varepsilon > 0$ such that $x \in (x-\varepsilon, x+\varepsilon) \subset G_{\lambda_x} \subset \bigcup_{\lambda\in\Lambda} G_\lambda$. This shows $\bigcup_{\lambda\in\Lambda} G_\lambda$ is open.

(c) If $x \in \bigcap_{k=1}^n G_k$, then $x \in G_k$ for $1 \le k \le n$. For each G_k there is an ε_k such that $(x-\varepsilon_k, x+\varepsilon_k) \subset G_k$. Let $\varepsilon = \min\{\varepsilon_k : 1 \le k \le n\}$. Then $(x-\varepsilon, x+\varepsilon) \subset G_k$ for $1 \le k \le n$, so $(x-\varepsilon, x+\varepsilon) \subset \bigcap_{k=1}^n G_k$. Therefore $\bigcap_{k=1}^n G_k$ is open. □

The word *finite* in part (c) of the theorem is important because the intersection of an infinite number of open sets need not be open. For example, let $G_n = (-1/n, 1/n)$ for $n \in \mathbb{N}$. Then each G_n is open, but $\bigcap_{n\in\mathbb{N}} G_n = \{0\}$ is not.

Applying DeMorgan's laws to the parts of Theorem 5.2 yields the next corollary.

Corollary 5.3. *Following are the basic properties of closed sets.*

(a) *Both $\emptyset$ and $\mathbb{R}$ are closed.*

(b) *If $\{F_\lambda : \lambda \in \Lambda\}$ is a collection of closed sets, then $\bigcap_{\lambda\in\Lambda} F_\lambda$ is closed.*

(c) *If $\{F_k : 1 \le k \le n\}$ is a finite collection of closed sets, then $\bigcup_{k=1}^n F_k$ is closed.*

Surprisingly, $\emptyset$ and $\mathbb{R}$ are both open and closed. They are the only subsets of $\mathbb{R}$ with this dual personality. Sets that are both open and closed are sometimes said to be *clopen*.

5.1.1 Topological Spaces

The preceding theorem provides the starting point for a fundamental area of mathematics called *topology*. The properties of the open sets of $\mathbb{R}$ motivated the following definition.

Definition 5.4. For X a set, not necessarily a subset of $\mathbb{R}$, let $\mathcal{T} \subset \mathcal{P}(X)$. The set $\mathcal{T}$ is called a *topology* on X if it satisfies the following three conditions.

(a) $X \in \mathcal{T}$ and $\emptyset \in \mathcal{T}$.

(b) The union of any collection of sets from $\mathcal{T}$ is also in $\mathcal{T}$.

(c) The intersection of any finite collection of sets from $\mathcal{T}$ is also in $\mathcal{T}$.

The pair $(X, \mathcal{T})$ is called a topological space. The elements of $\mathcal{T}$ are the *open* sets of the topological space. The *closed sets* of the topological space are those sets whose complements are open.

It is easy to see both $(X, \mathcal{P}(X))$ and $(X, \{X, \emptyset\})$ are topologies on X. The former is called the *discrete* topology and the latter the *trivial* topology. Neither of these topologies is very interesting.

If $\mathcal{O} = \{G \subset \mathbb{R} : G \text{ is open}\}$, then Theorem 5.2 shows $(\mathbb{R}, \mathcal{O})$ is a topological space and $\mathcal{O}$ is called the *standard topology* on $\mathbb{R}$. While the standard topology is the most widely used topology, there are many other possible topologies on $\mathbb{R}$. For example, $\mathcal{R} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\mathbb{R}, \emptyset\}$ is a topology on $\mathbb{R}$ called the *right ray* topology. The collection $\mathcal{F} = \{S \subset \mathbb{R} : S^c \text{ is finite}\} \cup \{\emptyset\}$ is called the *finite complement* topology. The study of topologies is a huge area, further discussion of which would take us too far afield. There are many fine books on the subject ([18]) to which one can refer.

5.1.2 Limit Points and Closure

Definition 5.5. x_0 is a *limit point*[1] of $S \subset \mathbb{R}$ if for every $\varepsilon > 0$,

$$(x_0 - \varepsilon, x_0 + \varepsilon) \cap S \setminus \{x_0\} \neq \emptyset.$$

The *derived set* of S is

$$S' = \{x : x \text{ is a limit point of } S\}.$$

A point $x_0 \in S \setminus S'$ is an *isolated point* of S.

[1]This use of the term *limit point* is not universal. Some authors use the term *accumulation point*. Others use *condensation point*, although this is more often used for those cases when every neighborhood of x_0 intersects S in an uncountable set.

Notice that limit points of S need not be elements of S, but isolated points of S must be elements of S. In a sense, limit points and isolated points are at opposite extremes. The definitions can be restated as follows:

$$x_0 \text{ is a limit point of } S \text{ iff } \forall\varepsilon > 0\, (S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \neq \emptyset)$$
$$x_0 \text{ is an isolated point of } S \text{ iff } \exists\varepsilon > 0\, (S \cap (x_0 - \varepsilon, x_0 + \varepsilon) = \{x_0\})$$

Example 5.1. If $S = (0, 1]$, then $S' = [0, 1]$ and S has no isolated points.

Example 5.2. If $T = \{1/n : n \in \mathbb{Z} \setminus \{0\}\}$, then $T' = \{0\}$ and all points of T are isolated points of T.

Theorem 5.6. *x_0 is a limit point of S iff there is a sequence $x_n \in S \setminus \{x_0\}$ such that $x_n \to x_0$.*

Proof. ($\Rightarrow$) For each $n \in \mathbb{N}$ choose $x_n \in S \cap (x_0 - 1/n, x_0 + 1/n) \setminus \{x_0\}$. Then $|x_n - x_0| < 1/n$ for all $n \in \mathbb{N}$, so $x_n \to x_0$.

($\Leftarrow$) Suppose x_n is a sequence from $x_n \in S \setminus \{x_0\}$ converging to x_0. If $\varepsilon > 0$, the definition of convergence for a sequence yields an $N \in \mathbb{N}$ such that whenever $n \geq N$, then $x_n \in S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\}$. This shows $S \cap (x_0 - \varepsilon, x_0 + \varepsilon) \setminus \{x_0\} \neq \emptyset$, and x_0 must be a limit point of S. □

There is some common terminology making much of this easier to state. If $x_0 \in \mathbb{R}$ and G is an open set containing x_0, then G is called a *neighborhood* of x_0. The observations given above can be restated in terms of neighborhoods.

Corollary 5.7. *Let $S \subset \mathbb{R}$.*

(a) *x_0 is a limit point of S iff every neighborhood of x_0 contains an infinite number of points from S.*

(b) *$x_0 \in S$ is an isolated point of S iff there is a neighborhood of x_0 containing only a finite number of points from S.*

Following is a generalization of Theorem 3.16.

Theorem 5.8 (Bolzano-Weierstrass Theorem)**.** *If $S \subset \mathbb{R}$ is bounded and infinite, then $S' \neq \emptyset$.*

Proof. Since S is infinite, we can choose a sequence, x_n, of distinct points from S. Because S is bounded, so is the sequence x_n. The Bolzano-Weierstrass Theorem (Theorem 3.16) implies x_n has a convergent subsequence $y_n \to y$. Theorem 5.6 implies $y \in S'$. □

Theorem 5.9. *A set $S \subset \mathbb{R}$ is closed iff it contains all its limit points.*

Proof. ($\Rightarrow$) Suppose S is closed and x_0 is a limit point of S. If $x_0 \notin S$, then S^c being open implies the existence of $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \cap S = \emptyset$. This contradicts Corollary 5.7(a). Therefore, $x_0 \in S$, and S contains all its limit points.

($\Leftarrow$) Since S contains all its limit points, if $x_0 \notin S$, Corollary 5.7 implies there must exist an $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \cap S = \emptyset$. It follows from this that S^c is open. Therefore S is closed. □

Definition 5.10. The *closure* of a set S is the set $\overline{S} = S \cup S'$.

For the set S of Example 5.1, $\overline{S} = [0, 1]$. In Example 5.2, $\overline{T} = \{1/n : n \in \mathbb{Z} \setminus \{0\}\} \cup \{0\}$. According to Theorem 5.9, the closure of any set is a closed set. A useful way to think about this is that $\overline{S}$ is the smallest closed set containing S. This is made more precise in Exercise 2.

5.1.3 Nested Sets

Definition 5.11. A collection of sets $\{S_n : n \in \mathbb{N}\}$ is said to be *nested*, if $S_{n+1} \subset S_n$ for all $n \in \mathbb{N}$.

Theorem 5.12. *If $\{F_n : n \in \mathbb{N}\}$ is a nested collection of nonempty closed and bounded sets, then $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$. Moreover, if $\operatorname{lub} F_n - \operatorname{glb} F_n \to 0$, then there is y such that $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset = \{y\}$.*

Proof. Form a sequence x_n by choosing $x_n \in F_n$ for each $n \in \mathbb{N}$. Since the F_n are nested, $\{x_n : n \in \mathbb{N}\} \subset F_1$, and the boundedness of F_1 implies x_n is a bounded sequence. An application of Theorem 3.16 yields a subsequence y_n of x_n such that $y_n \to y$. It suffices to prove $y \in F_n$ for all $n \in \mathbb{N}$.

To do this, fix $n_0 \in \mathbb{N}$. Because y_n is a subsequence of x_n and $x_{n_0} \in F_{n_0}$, it is easy to see $y_n \in F_{n_0}$ for all $n \geq n_0$. Using the fact that $y_n \to y$, we see $y \in F'_{n_0}$. Since F_{n_0} is closed, Theorem 5.9 shows $y \in F_{n_0}$.

The proof of the second part of the theorem is left as Exercise 19 □

Theorem 5.12 is most often used in the following special case.

Corollary 5.13 (Nested Interval Theorem)**.** *If $\{I_n = [a_n, b_n] : n \in \mathbb{N}\}$ is a nested collection of closed intervals such that $\lim_{n\to\infty}(b_n - a_n) = 0$, then there is an $x \in \mathbb{R}$ such that $\bigcap_{n\in\mathbb{N}} I_n = \{x\}$.*

Example 5.3. If $I_n = (0, 1/n]$ for all $n \in \mathbb{N}$, then the collection $\{I_n : n \in \mathbb{N}\}$ is nested, but $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$. This shows the assumption that the intervals be closed in the Nested Interval Theorem is necessary.

Example 5.4. If $I_n = [n, \infty)$ then the collection $\{I_n : n \in \mathbb{N}\}$ is nested, but $\bigcap_{n\in\mathbb{N}} I_n = \emptyset$. This shows the assumption that the lengths of the intervals be bounded is necessary.

5.2 Covering Properties and Compactness on $\mathbb{R}$

5.2.1 Open Covers

Definition 5.14. Let $S \subset \mathbb{R}$. A collection of open sets, $\mathcal{O} = \{G_\lambda : \lambda \in \Lambda\}$, is an *open cover* of S, if $S \subset \bigcup_{G\in\mathcal{O}} G$. If $\mathcal{O}' \subset \mathcal{O}$ is also an open cover of S, then $\mathcal{O}'$ is an *open subcover of S from $\mathcal{O}$.*

Example 5.5. Let $S = (0, 1)$ and $\mathcal{O} = \{(1/n, 1) : n \in \mathbb{N}\}$. We claim that $\mathcal{O}$ is an open cover of S. To prove this, let $x \in (0, 1)$. Choose $n_0 \in \mathbb{N}$ such that $1/n_0 < x$. Then

$$x \in (1/n_0, 1) \subset \bigcup_{n\in\mathbb{N}} (1/n, 1) = \bigcup_{G\in\mathcal{O}} G.$$

Since x is an arbitrary element of $(0, 1)$, it follows that $(0, 1) = \bigcup_{G\in\mathcal{O}} G$.

Suppose $\mathcal{O}'$ is any infinite subset of $\mathcal{O}$ and $x \in (0, 1)$. Since $\mathcal{O}'$ is infinite, there exists an $n \in \mathbb{N}$ such that $x \in (1/n, 1) \in \mathcal{O}'$. The rest of the proof proceeds as above.

On the other hand, if $\mathcal{O}'$ is a finite subset of $\mathcal{O}$, then let $M = \max\{n : (1/n, 1) \in \mathcal{O}'\}$. If $0 < x < 1/M$, it is clear that $x \notin \bigcup_{G\in\mathcal{O}'} G$, so $\mathcal{O}'$ is not an open cover of $(0, 1)$.

Example 5.6. Let $T = [0, 1)$ and $0 < \varepsilon < 1$. If

$$\mathcal{O} = \{(1/n, 1) : n \in \mathbb{N}\} \cup \{(-\varepsilon, \varepsilon)\},$$

then $\mathcal{O}$ is an open cover of T.

It is evident that any open subcover of T from $\mathcal{O}$ must contain $(-\varepsilon, \varepsilon)$, because that is the only element of $\mathcal{O}$ which contains 0. Choose $n \in \mathbb{N}$ such that $1/n < \varepsilon$. Then $\mathcal{O}' = \{(-\varepsilon, \varepsilon), (1/n, 1)\}$ is an open subcover of T from $\mathcal{O}$ which contains only two elements.

Theorem 5.15 (Lindelöf Property). [3] *If $S \subset \mathbb{R}$ and $\mathcal{O}$ is any open cover of S, then $\mathcal{O}$ contains a subcover with a countable number of elements.*

[2] March 20, 2023

[3] Ernst Leonard Lindelöf (1870–1946) was a Finnish mathematician.

Proof. Let $\mathcal{O} = \{G_\lambda : \lambda \in \Lambda\}$ be an open cover of S. Since $\mathcal{O}$ is an open cover of S, for each $x \in S$ there is a $\lambda_x \in \Lambda$ and numbers $p_x, q_x \in \mathbb{Q}$ satisfying $x \in (p_x, q_x) \subset G_{\lambda_x} \in \mathcal{O}$. The collection $\mathcal{T} = \{(p_x, q_x) : x \in S\}$ is an open cover of S.

Thinking of the collection $\mathcal{T} = \{(p_x, q_x) : x \in S\}$ as a set of ordered pairs of rational numbers, it is seen that

$$\operatorname{card}(\mathcal{T}) \leq \operatorname{card}(\mathbb{Q} \times \mathbb{Q}) = \aleph_0,$$

where Theorems 2.27 and 1.20 were used.

For each interval $I \in \mathcal{T}$, choose a $\lambda_I \in \Lambda$ such that $I \subset G_{\lambda_I}$. Then

$$S \subset \bigcup_{I \in \mathcal{T}} I \subset \bigcup_{I \in \mathcal{T}} G_{\lambda_I}$$

shows $\mathcal{O}' = \{G_{\lambda_I} : I \in \mathcal{T}\} \subset \mathcal{O}$ is an open subcover of S from $\mathcal{O}$. Also, $\operatorname{card}(\mathcal{O}') \leq \operatorname{card}(\mathcal{T}) \leq \aleph_0$, so $\mathcal{O}'$ is a countable open subcover of S from $\mathcal{O}$. □

Lindelöf's Theorem implies open sets have a simple structure, as seen in the following corollary.

Corollary 5.16. *Any open subset of $\mathbb{R}$ can be written as a countable union of pairwise disjoint open intervals.*

Proof. Let G be open in $\mathbb{R}$. For $x \in G$ let $\alpha_x = \operatorname{glb}\{y : (y, x] \subset G\}$ and $\beta_x = \operatorname{lub}\{y : [x, y) \subset G\}$. The fact that G is open implies $\alpha_x < x < \beta_x$. Define $I_x = (\alpha_x, \beta_x)$.

Then $I_x \subset G$. To see this, suppose $x < w < \beta_x$. Choose $y \in (w, \beta_x)$. The definition of β_x guarantees $w \in (x, y) \subset G$. Similarly, if $\alpha_x < w < x$, it follows that $w \in G$.

This shows $\mathcal{O} = \{I_x : x \in G\}$ has the property that $G = \bigcup_{x \in G} I_x$.

Suppose $x, y \in G$, $x < y$ and $I_x \cap I_y \neq \emptyset$. In this case, there must be a $w \in (x, y)$ such that $w \in I_x \cap I_y$. We know from above that both $[x, w] \subset G$ and $[w, y] \subset G$, so $[x, y] \subset G$. It follows that $\alpha_x = \alpha_y < x < y < \beta_x = \beta_y$ and $I_x = I_y$.

From this we conclude $\mathcal{O}$ consists of pairwise disjoint open intervals.

To finish, apply Theorem 5.2.1 to extract a countable subcover from $\mathcal{O}$. □

Corollary 5.16 can also be proved by a different strategy. Instead of using Theorem 5.2.1 to extract a countable subcover, we could just choose one rational number from each interval in the cover. The pairwise disjointness of the intervals in the cover guarantee this will give a bijection between $\mathcal{O}$ and a subset of $\mathbb{Q}$. This method has the advantage of showing $\mathcal{O}$ itself is countable from the start.

5.2.2 Compact Sets

There is a class of sets for which the conclusion of Lindelöf's theorem can be strengthened.

Definition 5.17. An open cover $\mathcal{O}$ of a set S is a *finite cover*, if $\mathcal{O}$ has only a finite number of elements. The definition of a *finite subcover* is analogous.

Definition 5.18. A set $K \subset \mathbb{R}$ is *compact*, if every open cover of K contains a finite subcover.

Theorem 5.19 (Heine-Borel)**.** *A set $K \subset \mathbb{R}$ is compact iff it is closed and bounded.*

Proof. ($\Rightarrow$) Suppose K is unbounded. The collection $\mathcal{O} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of K. If $\mathcal{O}'$ is any finite subset of $\mathcal{O}$, then $\bigcup_{G \in \mathcal{O}'} G$ is a bounded set and cannot cover the unbounded set K. This shows K cannot be compact, and every compact set must be bounded.

Suppose K is not closed. According to Theorem 5.9, there is a limit point x of K such that $x \notin K$. Define $\mathcal{O} = \{[x - 1/n, x + 1/n]^c : n \in \mathbb{N}\}$. Then $\mathcal{O}$ is a collection of open sets and $K \subset \bigcup_{G \in \mathcal{O}} G = \mathbb{R} \setminus \{x\}$. Let $\mathcal{O}' = \{[x - 1/n_i, x + 1/n_i]^c : 1 \leq i \leq N\}$ be a finite subset of $\mathcal{O}$ and $M = \max\{n_i : 1 \leq i \leq N\}$. Since x is a limit point of K, there is a $y \in K \cap (x - 1/M, x + 1/M)$. Clearly, $y \notin \bigcup_{G \in \mathcal{O}'} G = [x - 1/M, x + 1/M]^c$, so $\mathcal{O}'$ cannot cover K. This shows every compact set must be closed.

($\Leftarrow$) Let K be closed and bounded and let $\mathcal{O}$ be an open cover of K. Applying Lindelöf's Theorem 5.2.1, if necessary, we can assume $\mathcal{O}$ is countable. Thus, $\mathcal{O} = \{G_n : n \in \mathbb{N}\}$.

For each $n \in \mathbb{N}$, define

$$F_n = K \setminus \bigcup_{i=1}^{n} G_i = K \cap \bigcap_{i=1}^{n} G_i^c.$$

Then F_n is a sequence of nested, bounded and closed subsets of K. Since $\mathcal{O}$ covers K, it follows that

$$\bigcap_{n \in \mathbb{N}} F_n \subset K \setminus \bigcup_{n \in \mathbb{N}} G_n = \emptyset.$$

According to the Corollary 5.12, the only way this can happen is if $F_n = \emptyset$ for some $n \in \mathbb{N}$. Then $K \subset \bigcup_{i=1}^{n} G_i$, and $\mathcal{O}' = \{G_i : 1 \leq i \leq n\}$ is a finite subcover of K from $\mathcal{O}$. □

Compactness shows up in several different, but equivalent ways on $\mathbb{R}$. We've already seen several of them, but their equivalence is not obvious. The following theorem shows a few of the most common manifestations of compactness.

Theorem 5.20. *Let $K \subset \mathbb{R}$. The following statements imply each other.*

(*a*) *K is compact.*

(*b*) *K is closed and bounded.*

(*c*) *Every infinite subset of K has a limit point in K.*

(*d*) *Every sequence $\{a_n : n \in \mathbb{N}\} \subset K$ has a subsequence converging to an element of K.*

(*e*) *If F_n is a nested sequence of nonempty relatively closed subsets of K, then $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$.*

Proof. (a) $\iff$ (b) is the Heine-Borel Theorem, Theorem 5.19.

That (b)$\Rightarrow$(c) is the Bolzano-Weierstrass Theorem, Theorem 5.8.

(c)$\Rightarrow$(d) is contained in the sequence version of the Bolzano-Weierstrass Theorem, Theorem 3.16.

(d)$\Rightarrow$(e) Let F_n be as in (e). For each $n \in \mathbb{N}$, choose $a_n \in F_n \cap K$. By assumption, a_n has a convergent subsequence $b_n \to b \in K$. Each F_n contains a tail of the sequence b_n, so $b \in F_n' \subset F_n$ for each n. Therefore, $b \in \bigcap_{n\in\mathbb{N}} F_n$, and (e) follows.

(e)$\Rightarrow$(b). Suppose K is such that (e) is true.

Let $F_n = ((-\infty, -n] \cup [n, \infty))$. Then F_n is a sequence of closed sets such that $\bigcap_{n\in\mathbb{N}} F_n = \emptyset$. Since $F_n \cap K \neq \emptyset$, $\forall n \in \mathbb{N}$, a contradiction of (e) is evident. Therefore, K must be bounded.

If K is not closed, then there must be a limit point x of K such that $x \notin K$. Define a sequence of closed and nested intervals by $F_n = [x - 1/n, x + 1/n]$. Then $\bigcap_{n\in\mathbb{N}} F_n = \{x\}$, and $F_n \cap K \neq \emptyset$. This contradiction of (e) shows that K must be closed. □

These various ways of looking at compactness have been given different names by topologists. Property (c) is called *limit point compactness* and (d) is called *sequential compactness*. There are topological spaces in which various of the equivalences do not hold.

5.3 Relative Topologies and Connectedness

5.3.1 Relative Topologies

Another useful topological notion is that of a relative or subspace topology. In our case, this amounts to using the standard topology on $\mathbb{R}$ to induce a topology on a subset of $\mathbb{R}$. The definition is as follows.

Definition 5.21. Let $X \subset \mathbb{R}$. The set $S \subset X$ is *relatively open* in X, if there is a set G, open in $\mathbb{R}$, such that $S = G \cap X$. The set $T \subset X$ is *relatively closed* in X, if there is a set F, closed in $\mathbb{R}$, such that $S = F \cap X$. (If there is no chance for confusion, the simpler terminology *open in* X and *closed in* X is usually used.)

It is left as exercises to show that if $X \subset \mathbb{R}$ and $\mathcal{S}$ consists of all relatively open subsets of X, then $(X, \mathcal{S})$ is a topological space and T is relatively closed in X, if $X \setminus T \in \mathcal{S}$. (See Exercises 12 and 13.)

Example 5.7. Let $X = [0, 1]$. The subsets $[0, 1/2) = X \cap (-1, 1/2)$ and $(1/4, 1] = X \cap (1/4, 2)$ are both relatively open in X.

Example 5.8. If $X = \mathbb{Q}$, then $\{x \in \mathbb{Q} : -\sqrt{2} < x < \sqrt{2}\} = (-\sqrt{2}, \sqrt{2}) \cap \mathbb{Q} = [-\sqrt{2}, \sqrt{2}] \cap \mathbb{Q}$ is both open and closed (clopen) relative to $\mathbb{Q}$.

5.3.2 Connected Sets

One place where the relative topologies are useful is in relation to the following definition.

Definition 5.22. A set $S \subset \mathbb{R}$ is *disconnected* if there are two open intervals U and V such that $U \cap V = \emptyset$, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$ and $S \subset U \cup V$. Otherwise, it is *connected*. The sets $U \cap S$ and $V \cap S$ are said to be a separation of S.

In other words, S is disconnected if it can be written as the union of two disjoint and nonempty sets that are both relatively open in S. Since both these sets are complements of each other relative to S, they are both *clopen* in S. This, in turn, implies S is disconnected if and only if it has a proper relatively clopen subset.

Example 5.9. Let $S = \{x\}$ be a set containing a single point. S is connected because there cannot exist nonempty disjoint open sets U and V such that $S \cap U \neq \emptyset$ and $S \cap V \neq \emptyset$. The same argument shows that $\emptyset$ is connected.

Example 5.10. If $S = [-1, 0) \cup (0, 1]$, then $U = (-2, 0)$ and $V = (0, 2)$ are open sets such that $U \cap V = \emptyset$, $U \cap S \neq \emptyset$, $V \cap S \neq \emptyset$ and $S \subset U \cup V$. This shows S is disconnected.

Example 5.11. The sets $U = (-\infty, \sqrt{2})$ and $V = (\sqrt{2}, \infty)$ are open sets such that $U \cap V = \emptyset$, $U \cap \mathbb{Q} \neq \emptyset$, $V \cap \mathbb{Q} \neq \emptyset$ and $\mathbb{Q} \subset U \cup V = \mathbb{R} \setminus \{\sqrt{2}\}$. This shows $\mathbb{Q}$ is disconnected. In fact, the only connected subsets of $\mathbb{Q}$ are single points. Sets with this property are often called *totally disconnected*.

The notion of connectedness is not really very interesting on $\mathbb{R}$ because the connected sets are exactly what one would expect. It becomes more complicated in higher dimensional spaces. The following theorem is not surprising.

Theorem 5.23. *A nonempty set $S \subset \mathbb{R}$ is connected iff it is either a single point or an interval.*

Proof. ($\Rightarrow$) If S is not a single point or an interval, there must be numbers $r < s < t$ such that $r, t \in S$ and $s \notin S$. In this case, the sets $U = (-\infty, s)$ and $V = (s, \infty)$ are a disconnection of S.

($\Leftarrow$) It was shown in Example 5.9 that a set containing a single point is connected. So, assume S is an interval.

Suppose S is not connected with U and V forming a disconnection of S. Choose $u \in U \cap S$ and $v \in V \cap S$. There is no generality lost by assuming $u < v$, so that $[u, v] \subset S$.

Let $A = \{x : [u, x) \subset U\}$.

We claim $A \neq \emptyset$. To see this, use the fact that U is open to find $\varepsilon \in (0, v - u)$ such that $(u - \varepsilon, u + \varepsilon) \subset U$. Then $u < u + \varepsilon/2 < v$, so $u + \varepsilon/2 \in A$.

Define $w = \text{lub}\, A$.

Since $v \in V$ it is evident $u < w \leq v$ and $w \in S$.

If $w \in U$, then $u < w < v$ and there is $\varepsilon \in (0, v - w)$ such that $(w - \varepsilon, w + \varepsilon) \subset U$ and $[u, w + \varepsilon) \subset S$ because $w + \varepsilon < v$. This clearly contradicts the definition of w, so $w \notin U$.

If $w \in V$, then there is an $\varepsilon > 0$ such that $(w - \varepsilon, w] \subset V$. In particular, this shows $w = \text{lub}\, A \leq w - \varepsilon < w$. This contradiction forces the conclusion that $w \notin V$.

Now, putting all this together, we see $w \in S \subset U \cup V$ and $w \notin U \cup V$. This is a clear contradiction, so we're forced to conclude there is no separation of S. $\square$

5.4 More Small Sets

This is an advanced section that can be omitted.

We've already seen one way in which a subset of $\mathbb{R}$ can be considered small—if its cardinality is at most $\aleph_0$. Such sets are small in the set-theoretic sense. This section shows how sets can be considered small in the metric and topological senses.

5.4.1 Sets of Measure Zero

An interval is the only subset of $\mathbb{R}$ for which most people could immediately come up with some sort of measure—namely, its length. This idea of measuring a set by length can be generalized. For example, we know every open set can be written as a countable disjoint union of open intervals, so it is natural to assign the sum of the lengths of its component intervals as the measure of the set. Discounting some technical difficulties, such as components with infinite length, this is how the Lebesgue measure of an open set is defined. It is possible to assign a measure to more complicated sets, but we'll only address the special case of sets with measure zero, sometimes called *Lebesgue null sets.*

Definition 5.24. A set $S \subset \mathbb{R}$ has measure zero if given any $\varepsilon > 0$ there is a sequence (a_n, b_n) of open intervals such that

$$S \subset \bigcup_{n\in\mathbb{N}} (a_n, b_n) \text{ and } \sum_{n=1}^{\infty} (b_n - a_n) < \varepsilon.$$

Such sets are small in the metric sense.

Example 5.12. If $S = \{a\}$ contains only one point, then S has measure zero. To see this, let $\varepsilon > 0$. Note that $S \subset (a - \varepsilon/4, a + \varepsilon/4)$ and this single interval has length $\varepsilon/2 < \varepsilon$.

There are complicated sets of measure zero, as we'll see later. For now, we'll start with a simple theorem.

Theorem 5.25. *If $\{S_n : n \in \mathbb{N}\}$ is a countable collection of sets of measure zero, then $\bigcup_{n\in\mathbb{N}} S_n$ has measure zero.*

Proof. Let $\varepsilon > 0$. For each n, let $\{(a_{n,k}, b_{n,k}) : k \in \mathbb{N}\}$ be a collection of intervals such that

$$S_n \subset \bigcup_{k\in\mathbb{N}} (a_{n,k}, b_{n,k}) \text{ and } \sum_{k=1}^{\infty} (b_{n,k} - a_{n,k}) < \frac{\varepsilon}{2^n}.$$

Then

$$\bigcup_{n\in\mathbb{N}} S_n \subset \bigcup_{n\in\mathbb{N}} \bigcup_{k\in\mathbb{N}} (a_{n,k}, b_{n,k}) \text{ and } \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (b_{n,k} - a_{n,k}) < \sum_{n=1}^{\infty} \frac{\varepsilon}{2^n} = \varepsilon.$$

$\square$

Combining this with Example 5.12 gives the following corollary.

Corollary 5.26. *Every countable set has measure zero.*

The rational numbers is a large set in the sense that every interval contains a rational number. But we now see it is small in both the set theoretic and metric senses because it is countable and of measure zero.

Uncountable sets of measure zero are constructed in Section 5.4.3.

There is some standard terminology associated with sets of measure zero. If a property P is true, except on a set of measure zero, then it is often said "P is true *almost everywhere*" or "*almost every* point satisfies P." It is also said "P is true on a set of *full measure*." For example, "Almost every real number is irrational." or "The irrational numbers are a set of full measure."

5.4.2 Dense and Nowhere Dense Sets

We begin by considering a way that a set can be considered topologically large in an interval. If I is any interval, recall from Corollary 2.25 that $I \cap \mathbb{Q} \neq \emptyset$ and $I \cap \mathbb{Q}^c \neq \emptyset$. An immediate consequence of this is that every real number is a limit point of both $\mathbb{Q}$ and $\mathbb{Q}^c$. In this sense, the rational and irrational numbers are both uniformly distributed across the number line. This idea is generalized in the following definition.

Definition 5.27. Let $A \subset B \subset \mathbb{R}$. A is said to be *dense* in B, if $B \subset \overline{A}$.

Both the rational and irrational numbers are dense in every interval. Corollary 5.16 shows the rational and irrational numbers are dense in every open set. It's not hard to construct other sets dense in every interval. For example, the set of dyadic numbers, $\mathcal{D} = \{p/2^q : p, q \in \mathbb{Z}\}$, is dense in every interval — and dense in the rational numbers.

On the other hand, $\mathbb{Z}$ is not dense in any interval because it's closed and contains no interval. If $A \subset B$, where B is an open set, then A is not dense in B, if A contains any interval-sized gaps.

Theorem 5.28. *Let $A \subset B \subset \mathbb{R}$. A is dense in B iff whenever I is an open interval such that $I \cap B \neq \emptyset$, then $I \cap A \neq \emptyset$.*

Proof. ($\Rightarrow$) Assume there is an open interval I such that $I \cap B \neq \emptyset$ and $I \cap A = \emptyset$. If $x \in I \cap B$, then I is a neighborhood of x that does not intersect A. Definition 5.5 shows $x \notin A' \subset \overline{A}$, a contradiction of the assumption that $B \subset \overline{A}$. This contradiction implies that whenever $I \cap B \neq \emptyset$, then $I \cap A \neq \emptyset$.

($\Leftarrow$) If $x \in B \cap A = A$, then $x \in \overline{A}$. Assume $x \in B \setminus A$. By assumption, for each $n \in \mathbb{N}$, there is an $x_n \in (x - 1/n, x + 1/n) \cap A$. Since $x_n \to x$, this shows $x \in A' \subset \overline{A}$. It now follows that $B \subset \overline{A}$. $\square$

If $B \subset \mathbb{R}$ and I is an open interval with $I \cap B \neq \emptyset$, then $I \cap B$ is often called a *portion* of B. The previous theorem says that A is dense in B iff every portion of B intersects A.

If A being dense in B is thought of as A being a large subset of B, then perhaps when A is not dense in B, it can be thought of as a small subset. But, thinking of A as being small when it is not dense isn't quite so clear when it is noticed that A could still be dense in some portion of B, even if it isn't dense in B. To make A be a truly small subset of B in the topological sense, it should not be dense in any portion of B. The following definition gives a way to assure this is true.

Definition 5.29. Let $A \subset B \subset \mathbb{R}$. A is said to be *nowhere dense* in B if $B \setminus \overline{A}$ is dense in B.

The following theorem shows that a nowhere dense set is small in the sense mentioned above because it fails to be dense in any part of B.

Theorem 5.30. *Let $A \subset B \subset \mathbb{R}$. A is nowhere dense in B iff for every open interval I such that $I \cap B \neq \emptyset$, there is an open interval $J \subset I$ such that $J \cap B \neq \emptyset$ and $J \cap A = \emptyset$.*

Proof. ($\Rightarrow$) Let I be an open interval such that $I \cap B \neq \emptyset$. By assumption, $B \setminus \overline{A}$ is dense in B, so Theorem 5.28 implies $I \cap (B \setminus \overline{A}) \neq \emptyset$. If $x \in I \cap (B \setminus \overline{A})$, then there is an open interval J such that $x \in J \subset I$ and $J \cap \overline{A} = \emptyset$. Since $A \subset \overline{A}$, this J satisfies the theorem.

($\Leftarrow$) Let I be an open interval with $I \cap B \neq \emptyset$. By assumption, there is an open interval $J \subset I$ such that $J \cap A = \emptyset$. It follows that $J \cap \overline{A} = \emptyset$. Theorem 5.28 implies $B \setminus \overline{A}$ is dense in B. □

Example 5.13. Let G be an open set that is dense in $\mathbb{R}$. If I is any open interval, then Theorem 5.28 implies $I \cap G \neq \emptyset$. Because G is open, if $x \in I \cap G$, then there is an open interval J such that $x \in J \subset G$. Now, Theorem 5.30 shows G^c is nowhere dense.

The nowhere dense sets are topologically small in the following sense.

Theorem 5.31 (Baire)**.** *If I is an open interval, then I cannot be written as a countable union of nowhere dense sets.*

Proof. Let A_n be a sequence of nowhere dense subsets of I. According to Theorem 5.30, there is a bounded open interval $J_1 \subset I$ such that $J_1 \cap A_1 = \emptyset$. By shortening J_1 a bit at each end, if necessary, it may be assumed that $\overline{J_1} \cap A_1 = \emptyset$. Assume J_n has been chosen for some $n \in \mathbb{N}$. Applying Theorem 5.30 again, choose an open interval J_{n+1} as above so $J_{n+1} \subset J_n$ and $\overline{J_{n+1}} \cap A_{n+1} = \emptyset$. Corollary 5.12 shows

$$I \setminus \bigcup_{n\in\mathbb{N}} A_n \supset \bigcap_{n\in\mathbb{N}} \overline{J_n} \neq \emptyset$$

and the theorem follows. □

Theorem 5.31 is called the *Baire category theorem* because of the terminology introduced by René-Louis Baire in 1899.[4] He said a set was of the *first category*, if it could be written as a countable union of nowhere dense sets. An easy example of such a set is any countable set, which is a countable union of singletons. All other sets are of the *second category*.[5] Theorem 5.31 can be stated as "Any open interval is of the second category." Or, more generally, as "Any nonempty open set is of the second category."

A set is called a $\mathbf{G}_\delta$ set, if it is the countable intersection of open sets. It is called an $\mathbf{F}_\sigma$ set, if it is the countable union of closed sets. De Morgan's laws show that the complement of an $\mathbf{F}_\sigma$ set is a $\mathbf{G}_\delta$ set and vice versa. It's evident that any countable subset of $\mathbb{R}$ is an $\mathbf{F}_\sigma$ set, so $\mathbb{Q}$ is an $\mathbf{F}_\sigma$ set.

On the other hand, suppose $\mathbb{Q}$ is a $\mathbf{G}_\delta$ set. Then there is a sequence of open sets G_n such that $\mathbb{Q} = \bigcap_{n\in\mathbb{N}} G_n$. Since $\mathbb{Q}$ is dense, each G_n must be dense and Example 5.13 shows G_n^c is nowhere dense. From De Morgan's law, $\mathbb{R} = \mathbb{Q} \cup \bigcup_{n\in\mathbb{N}} G_n^c$, showing $\mathbb{R}$ is a first category set and violating the Baire category theorem. Therefore, $\mathbb{Q}$ is not a $\mathbf{G}_\delta$ set. An immediate consequence is that the irrational numbers are a $\mathbf{G}_\delta$ set, but not an $\mathbf{F}_\sigma$ set.

Essentially the same argument shows any countable subset of $\mathbb{R}$ is a first category set. The following protracted example shows there are uncountable sets of the first category.

5.4.3 The Cantor Middle-Thirds Set

One particularly interesting example of a nowhere dense set is the Cantor Middle-Thirds set, introduced by the German mathematician Georg Cantor in 1884.[6] It has many strange properties, only a few of which will be explored here.

[4]René-Louis Baire (1874-1932) was a French mathematician. He proved the Baire category theorem in his 1899 doctoral dissertation.

[5]Baire did not define any categories other than these two. Some authors call first category sets *meager* sets, so as not to make readers fruitlessly wait for definitions of third, fourth and fifth category sets.

[6]Cantor's original work [6] is reprinted with an English translation in Edgar's *Classics on Fractals* [12]. Cantor only mentions his eponymous set in passing and it had actually been presented earlier by others.

To start the construction of the Cantor Middle-Thirds set, let $C_0 = [0,1]$ and $C_1 = I_1 \setminus (1/3, 2/3) = [0,1/3] \cup [2/3,1]$. Remove the open middle thirds of the intervals comprising C_1, to get

$$C_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right].$$

Continuing in this way, if C_n consists of 2^n pairwise disjoint closed intervals each of length 3^{-n}, construct C_{n+1} by removing the open middle third from each of those closed intervals, leaving 2^{n+1} closed intervals each of length $3^{-(n+1)}$. This gives a nested sequence of closed sets C_n each consisting of 2^n closed intervals of length 3^{-n}. (See Figure 5.1.) The Cantor Middle-Thirds set is

$$C = \bigcap_{n \in \mathbb{N}} C_n.$$

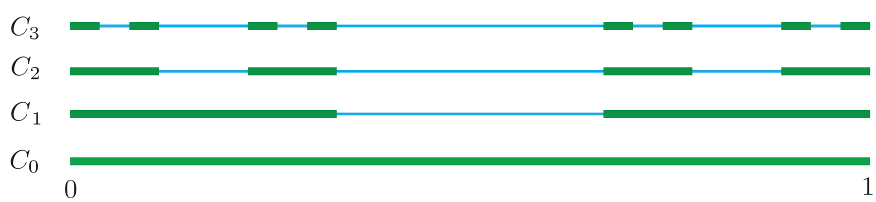


Figure 5.1: Shown here are the first few steps in the construction of the Cantor Middle-Thirds set.

Corollaries 5.3 and 5.12 show C is closed and nonempty. In fact, the latter is apparent because $\{0, 1/3, 2/3, 1\} \subset C_n$ for every n. At each step in the construction, 2^n open middle thirds, each of length $3^{-(n+1)}$ were removed from the intervals comprising C_n. The total length of the open intervals removed was

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = 1.$$

Because of this, Example 5.13 implies C is nowhere dense in $[0,1]$.

C is an example of a *perfect set*; i.e., a closed set all of whose points are limit points of itself. (See Exercise 26.) Any closed set without isolated points is perfect. The Cantor Middle-Thirds set is interesting because it is an example of a perfect set without any interior points. Many people call any bounded perfect set without interior points a Cantor set. Most of the time, when someone refers to *the* Cantor set, they mean C.

There is another way to view the Cantor set. Notice that at the nth stage of the construction, removing the middle thirds of the intervals comprising C_n removes those points whose base 3 representation contains the digit 1 in position $n+1$. Then,

$$C = \left\{ c = \sum_{n=1}^{\infty} \frac{c_n}{3^n} : c_n \in \{0,2\} \right\}. \tag{5.1}$$

So, C consists of all numbers $c \in [0,1]$ that can be written in base 3 without using the digit 1.[7]

If $c \in C$, then (5.1) shows $c = \sum_{n=1}^{\infty} c_n/3^n$ for some sequence c_n with range in $\{0,2\}$. Moreover, every such sequence corresponds to a unique element of C. Define $\phi : C \to [0,1]$ by

$$\phi(c) = \phi\left(\sum_{n=1}^{\infty} \frac{c_n}{3^n}\right) = \sum_{n=1}^{\infty} \frac{c_n/2}{2^n}. \tag{5.2}$$

Since c_n is a sequence from $\{0,2\}$, then $c_n/2$ is a sequence from $\{0,1\}$ and $\phi(c)$ can be considered the binary representation of a number in $[0,1]$. According to (5.1), it follows that ϕ is a surjection and

$$\phi(C) = \left\{ \sum_{n=1}^{\infty} \frac{c_n/2}{2^n} : c_n \in \{0,2\} \right\} = \left\{ \sum_{n=1}^{\infty} \frac{b_n}{2^n} : b_n \in \{0,1\} \right\} = [0,1].$$

Therefore, $\operatorname{card}(C) = \operatorname{card}([0,1]) > \aleph_0$.

The Cantor set is topologically small because it is nowhere dense and large from the set-theoretic viewpoint because it is uncountable.

The Cantor set is also a set of measure zero. To see this, let C_n be as in the construction of the Cantor set given above. Then $C \subset C_n$ and C_n consists of 2^n pairwise disjoint closed intervals each of length 3^{-n}. Their total length is $(2/3)^n$. Given $\varepsilon > 0$, choose $n \in \mathbb{N}$ so $(2/3)^n < \varepsilon/2$. Each of the closed intervals comprising C_n can be placed inside a slightly longer open interval so the sums of the lengths of the 2^n open intervals is less than ε.

5.5 Exercises

Exercise 5.1. If G is an open set and F is a closed set, then $G \setminus F$ is open and $F \setminus G$ is closed.

---

[7] Notice that $1 = \sum_{n=1}^{\infty} 2/3^n$, $1/3 = \sum_{n=2}^{\infty} 2/3^n$, etc.

Exercise 5.2. Let $S \subset \mathbb{R}$ and $\mathcal{F} = \{F : F \text{ is closed and } S \subset F\}$. Prove $\overline{S} = \bigcap_{F \in \mathcal{F}} F$. This proves that $\overline{S}$ is the smallest closed set containing S.

Exercise 5.3. Let S and T be subsets of $\mathbb{R}$. Prove or give a counterexample:

(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$, and

(b) $\overline{A \cap B} = \overline{A} \cap \overline{B}$

Exercise 5.4. If S is a finite subset of $\mathbb{R}$, then S is closed.

Exercise 5.5. For any sets $A, B \subset \mathbb{R}$, define

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

(a) If $X, Y \subset \mathbb{R}$, then $\overline{X} + \overline{Y} \subset \overline{X + Y}$.

(b) Find an example to show equality may not hold in the preceding statement.

Exercise 5.6. $\mathbb{Q}$ is neither open nor closed.

Exercise 5.7. A point x_0 is a boundary point of S if whenever U is a neighborhood of x_0, then $U \cap S \neq \emptyset$ and $U \cap S^c \neq \emptyset$. The set of all boundary points of S is denoted by ∂S. Prove that a set S is open iff $\partial S \cap S = \emptyset$.

Exercise 5.8. (a) Every closed set can be written as a countable intersection of open sets.

(b) Every open set can be written as a countable union of closed sets.

In other words, every closed set is a $\mathbf{G}_\delta$ set and every open set is an $\mathbf{F}_\sigma$ set.

Exercise 5.9. Find a sequence of open sets G_n such that $\bigcap_{n \in \mathbb{N}} G_n$ is neither open nor closed.

Exercise 5.10. A point x_0 is an interior point of S if there is an $\varepsilon > 0$ such that $(x_0 - \varepsilon, x_0 + \varepsilon) \subset S$. The set of all interior points of S is S°. An open set G is called *regular* if $G = (\overline{G})^\circ$. Find an open set that is not regular.

Exercise 5.11. Let $\mathcal{R} = \{(x, \infty) : x \in \mathbb{R}\}$ and $\mathcal{T} = \mathcal{R} \cup \{\mathbb{R}, \emptyset\}$. Prove that $(\mathbb{R}, \mathcal{T})$ is a topological space. This is called the *right ray topology* on $\mathbb{R}$.

Exercise 5.12. If $X \subset \mathbb{R}$ and $\mathcal{S}$ is the collection of all sets relatively open in X, then $(X, \mathcal{S})$ is a topological space.

Exercise 5.13. If $X \subset \mathbb{R}$ and G is an open set, then $X \setminus G$ is relatively closed in X.

Exercise 5.14. For any set S, let $\mathcal{F} = \{T \subset S : \text{card}\,(S \setminus T) \leq \aleph_0\} \cup \{\emptyset\}$. Then $(S, \mathcal{F})$ is a topological space. This is called the *finite complement* topology.

Exercise 5.15. An uncountable subset of $\mathbb{R}$ must have a limit point.

Exercise 5.16. If $S \subset \mathbb{R}$, then S' is closed.

Exercise 5.17. Prove that the set of accumulation points of any sequence is closed.

Exercise 5.18. Prove any closed set is the set of accumulation points for some sequence.

Exercise 5.19. Finish the proof of Theorem 5.12.

Exercise 5.20. If a_n is a sequence such that $a_n \to L$, then $\{a_n : n \in \mathbb{N}\} \cup \{L\}$ is compact.

Exercise 5.21. If F is closed and K is compact, then $F \cap K$ is compact.

Exercise 5.22. If $\{K_\alpha : \alpha \in A\}$ is a collection of compact sets, then $\bigcap_{\alpha \in A} K_\alpha$ is compact.

Exercise 5.23. Prove the union of a finite number of compact sets is compact. Give an example to show this need not be true for the union of an infinite number of compact sets.

Exercise 5.24. (a) Give an example of a set S such that S is disconnected, but $S \cup \{1\}$ is connected. (b) Prove that 1 must be a limit point of S.

Exercise 5.25. If K is compact and V is open with $K \subset V$, then there is an open set U such that $K \subset U \subset \overline{U} \subset V$.

Exercise 5.26. If C is the Cantor middle-thirds set, then $C = C'$.

Exercise 5.27. Prove that if $x \in \mathbb{R}$ and K is compact, then there is a $z \in K$ such that $|x - z| = \text{glb}\{|x - y| : y \in K\}$. Is z unique?

Exercise 5.28. If K is compact and $\mathcal{O}$ is an open cover of K, then there is an $\varepsilon > 0$ such that for all $x \in K$ there is a $G \in \mathcal{O}$ with $(x - \varepsilon, x + \varepsilon) \subset G$.

Exercise 5.29. Let $f : [a, b] \to \mathbb{R}$ be a function such that for every $x \in [a, b]$ there is a $\delta_x > 0$ such that f is bounded on $(x - \delta_x, x + \delta_x)$. Prove f is bounded.

Exercise 5.30. Is the function defined by (5.2) a bijection?

Exercise 5.31. If A is nowhere dense in an interval I, then $\overline{A}$ contains no interval.

Exercise 5.32. Use the Baire category theorem to show $\mathbb{R}$ is uncountable.

Exercise 5.33. If G is a dense $\mathbf{G}_\delta$ subset of $\mathbb{R}$, then G^c is a first category set.

Chapter 6

Limits of Functions

6.1 Basic Definitions

Definition 6.1. Let $D \subset \mathbb{R}$, x_0 be a limit point of D and $f : D \to \mathbb{R}$. The limit of $f(x)$ at x_0 is L, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that when $x \in D$ with $0 < |x - x_0| < \delta$, then $|f(x) - L| < \varepsilon$. When this is the case, we write $\lim_{x \to x_0} f(x) = L$.

It should be noted that the limit of f at x_0 is determined by the values of f *near* x_0 and not *at* x_0. In fact, f need not even be defined at x_0.

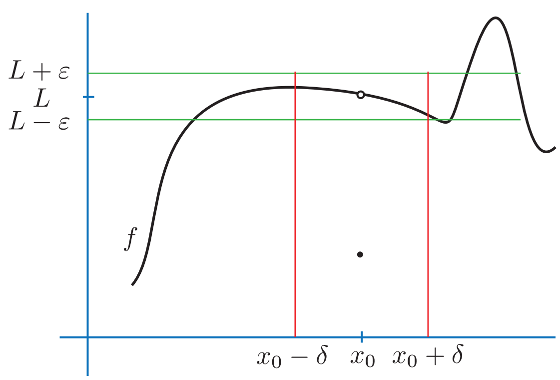


Figure 6.1: This figure shows a way to think about the limit. The graph of f must not leave the top or bottom of the box $(x_0 - \delta, x_0 + \delta) \times (L - \varepsilon, L + \varepsilon)$, except possibly the point $(x_0, f(x_0))$.

A useful way of rewording the definition is to say that $\lim_{x \to x_0} f(x) = L$ iff for every $\varepsilon > 0$ there is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$

[1] March 20, 2023

implies $f(x) \in (L-\varepsilon, L+\varepsilon)$. This can also be succinctly stated as

$$\forall \varepsilon > 0\, \exists \delta > 0\, (\, f\,(\,(x_0-\delta, x_0+\delta) \cap D \setminus \{x_0\}\,) \subset (L-\varepsilon, L+\varepsilon)\,)\,.$$

Example 6.1. If $f(x) = c$ is a constant function and $x_0 \in \mathbb{R}$, then for any positive numbers ε and δ,

$$x \in (x_0-\delta, x_0+\delta) \cap D \setminus \{x_0\} \Rightarrow |f(x) - c| = |c-c| = 0 < \varepsilon.$$

This shows the limit of every constant function exists at every point, and the limit is just the value of the function.

Example 6.2. Let $f(x) = x$, $x_0 \in \mathbb{R}$, and $\varepsilon = \delta > 0$. Then

$$x \in (x_0-\delta, x_0+\delta) \cap D \setminus \{x_0\} \Rightarrow |f(x) - x_0| = |x - x_0| < \delta = \varepsilon.$$

This shows that the identity function has a limit at every point and its limit is just the value of the function at that point.

Example 6.3. Let $f(x) = \frac{2x^2-8}{x-2}$. In this case, the implied domain of f is $D = \mathbb{R} \setminus \{2\}$. We claim that $\lim_{x\to 2} f(x) = 8$.

To see this, let $\varepsilon > 0$ and choose $\delta \in (0, \varepsilon/2)$. If $0 < |x-2| < \delta$, then

$$|f(x) - 8| = \left| \frac{2x^2-8}{x-2} - 8 \right| = |2(x+2) - 8| = 2|x-2| < \varepsilon.$$

As seen in Figure 6.2, the graph of f is just the graph of $y = 2x+4$ with a hole at $x = 2$, so this limit is obvious.

Example 6.4. Let $f(x) = \sqrt{x+1}$. Then the implied domain of f is $D = [-1, \infty)$. We claim that $\lim_{x\to -1} f(x) = 0$.

To see this, let $\varepsilon > 0$ and choose $\delta \in (0, \varepsilon^2)$. If $0 < x - (-1) = x+1 < \delta$, then

$$|f(x) - 0| = \sqrt{x+1} < \sqrt{\delta} < \sqrt{\varepsilon^2} = \varepsilon.$$

Example 6.5. If $f(x) = |x|/x$ for $x \neq 0$, then $\lim_{x\to 0} f(x)$ does not exist. (See Figure 6.3.) To see this, suppose $\lim_{x\to 0} f(x) = L$, $\varepsilon = 1$ and $\delta > 0$. If $L \geq 0$ and $-\delta < x < 0$, then $f(x) = -1 < L - \varepsilon$. If $L < 0$ and $0 < x < \delta$, then $f(x) = 1 > L + \varepsilon$. These inequalities show for any L and every $\delta > 0$, there is an x with $0 < |x| < \delta$ such that $|f(x) - L| > \varepsilon$.

There is an obvious similarity between the definitions of the limit of a sequence and the limit of a function. The following theorem makes this similarity explicit, and gives another way to prove facts about limits of functions.

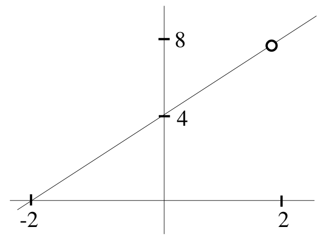


Figure 6.2: The function from Example 6.3. Note that the graph is a line with one "hole" in it.

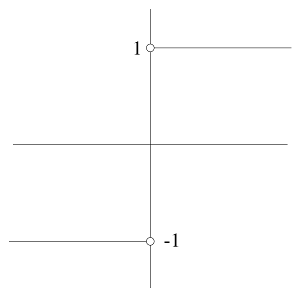


Figure 6.3: The function $f(x) = |x|/x$ from Example 6.5.

Theorem 6.2. *Let $f : D \to \mathbb{R}$ and x_0 be a limit point of D. $\lim_{x\to x_0} f(x) = L$ iff whenever x_n is a sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$, then $f(x_n) \to L$.*

Proof. ($\Rightarrow$) Suppose $\lim_{x\to x_0} f(x) = L$ and x_n is a sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$. Let $\varepsilon > 0$. There exists a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$. Since $x_n \to x_0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $0 < |x_n - x_0| < \delta$. In this case, $|f(x_n) - L| < \varepsilon$, showing $f(x_n) \to L$.

($\Leftarrow$) Suppose whenever x_n is a sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$, then $f(x_n) \to L$, but $\lim_{x\to x_0} f(x) \neq L$. Then there exists an $\varepsilon > 0$ such that for all $\delta > 0$ there is an $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$ such that $|f(x) - L| \geq \varepsilon$. In particular, for each $n \in \mathbb{N}$, there must exist $x_n \in (x_0 - 1/n, x_0 + 1/n) \cap D \setminus \{x_0\}$ such that $|f(x_n) - L| \geq \varepsilon$. Since $x_n \to x_0$, this is a contradiction. Therefore, $\lim_{x\to x_0} f(x) = L$. $\square$

Theorem 6.2 is often used to show a limit doesn't exist. Suppose we want to show $\lim_{x\to x_0} f(x)$ doesn't exist. There are two strategies: find a sequence $x_n \to x_0$ such that $f(x_n)$ has no limit; or, find two sequences $y_n \to x_0$ and $z_n \to x_0$ such that $f(y_n)$ and $f(z_n)$ converge to different limits. Either way, the theorem shows $\lim_{x\to x_0} f(x)$ fails to exist.

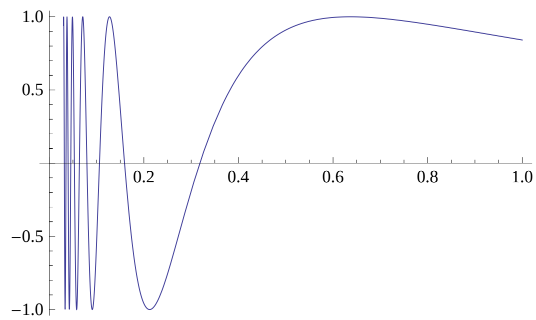


Figure 6.4: This is the function from Example 6.6. The graph shown here is on the interval $[0.03, 1]$. There are an infinite number of oscillations from -1 to 1 on any open interval containing the origin.

In Example 6.5, we could choose $x_n = (-1)^n/n$ so $f(x_n)$ oscillates between -1 and 1. Or, we could choose $y_n = 1/n$ and $z_n = -1/n$ so so both sequences converge to 0, $f(x_n) \to 1$ and $f(z_n) \to -1$.

Example 6.6. Let $f(x) = \sin(1/x)$, $a_n = \frac{1}{n\pi}$ and $b_n = \frac{2}{(4n+1)\pi}$. Then $a_n \downarrow 0$, $b_n \downarrow 0$, $f(a_n) = 0$ and $f(b_n) = 1$ for all $n \in \mathbb{N}$. An application of Theorem 6.2 shows $\lim_{x\to 0} f(x)$ does not exist. (See Figure 6.4.)

Theorem 6.3 (Squeeze Theorem)**.** *Suppose f, g and h are all functions defined on $D \subset \mathbb{R}$ with $f(x) \le g(x) \le h(x)$ for all $x \in D$. If x_0 is a limit point of D and $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L$, then $\lim_{x\to x_0} g(x) = L$.*

Proof. Let x_n be any sequence from $D \setminus \{x_0\}$ such that $x_n \to x_0$. According to Theorem 6.2, both $f(x_n) \to L$ and $h(x_n) \to L$. Since $f(x_n) \le g(x_n) \le h(x_n)$, an application of the Sandwich Theorem for sequences shows $g(x_n) \to L$. Now, another use of Theorem 6.2 shows $\lim_{x\to x_0} g(x) = L$. $\square$

Example 6.7. Let $f(x) = x\sin(1/x)$. Since $-1 \le \sin(1/x) \le 1$ when $x \ne 0$, we see that $-|x| \le x\sin(1/x) \le |x|$ for $x \ne 0$. Since $\lim_{x\to 0} |x| = 0$, Theorem 6.3 implies $\lim_{x\to 0} x\sin(1/x) = 0$. (See Figure 6.5.)

Theorem 6.4. *Suppose $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$ and x_0 is a limit point of D. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = M$, then*

(a) $\lim_{x\to x_0}(f+g)(x) = L + M$,

(b) $\lim_{x\to x_0}(af)(x) = aL$, $\forall a \in \mathbb{R}$,

(c) $\lim_{x\to x_0}(fg)(x) = LM$, *and*

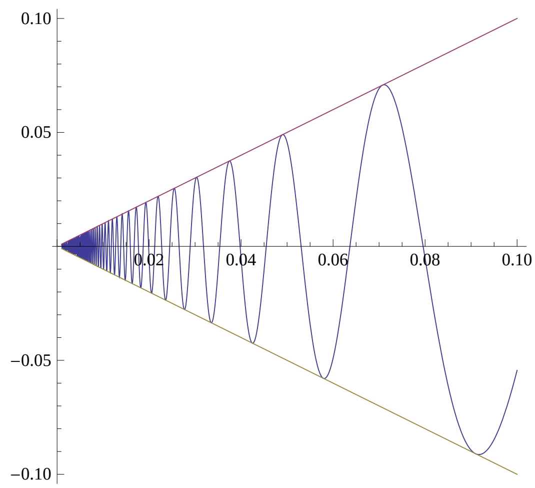


Figure 6.5: This is the function from Example 6.7. The bounding lines $y = x$ and $y = -x$ are also shown. There are an infinite number of oscillations between $-x$ and x on any open interval containing the origin.

(d) $\lim_{x \to x_0}(1/f)(x) = 1/L$, *as long as* $L \neq 0$.

Proof. Suppose a_n is a sequence from $D \setminus \{x_0\}$ converging to x_0. Then Theorem 6.2 implies $f(a_n) \to L$ and $g(a_n) \to M$. (a)-(d) follow at once from the corresponding properties for sequences. □

Example 6.8. Let $f(x) = 3x + 2$. If $g_1(x) = 3$, $g_2(x) = x$ and $g_3(x) = 2$, then $f(x) = g_1(x)g_2(x) + g_3(x)$. Examples 6.1 and 6.2 along with parts (a) and (c) of Theorem 6.4 immediately show that for every $x \in \mathbb{R}$, $\lim_{x \to x_0} f(x) = f(x_0)$.

In the same manner as Example 6.8, it can be shown for every rational function $f(x)$, that $\lim_{x \to x_0} f(x) = f(x_0)$ whenever x_0 is in the domain of f.

6.2 Unilateral Limits

Definition 6.5. Let $f : D \to \mathbb{R}$ and x_0 be a limit point of $(-\infty, x_0) \cap D$. f has L as its *left-hand limit* at x_0 if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $f((x_0 - \delta, x_0) \cap D) \subset (L - \varepsilon, L + \varepsilon)$. In this case, we write $\lim_{x \uparrow x_0} f(x) = L$.

Let $f : D \to \mathbb{R}$ and x_0 be a limit point of $D \cap (x_0, \infty)$. f has L as its *right-hand limit* at x_0 if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $f(D \cap (x_0, x_0 + \delta)) \subset (L - \varepsilon, L + \varepsilon)$. In this case, we write $\lim_{x \downarrow x_0} f(x) = L$.[2]

[2]Calculus books often use the notation $\lim_{x \uparrow x_0} f(x) = \lim_{x \to x_0-} f(x)$ and $\lim_{x \downarrow x_0} f(x) = \lim_{x \to x_0+} f(x)$.

These are called the *unilateral* or *one-sided* limits of f at x_0. When they are different, the graph of f is often said to have a "jump" at x_0, as in the following example.

Example 6.9. As in Example 6.5, let $f(x) = |x|/x$. Then $\lim_{x\downarrow 0} f(x) = 1$ and $\lim_{x\uparrow 0} f(x) = -1$. (See Figure 6.3.)

In parallel with Theorem 6.2, the one-sided limits can also be reformulated in terms of sequences.

Theorem 6.6. *Let $f : D \to \mathbb{R}$ and x_0.*

(a) *Let x_0 be a limit point of $D \cap (x_0, \infty)$. $\lim_{x\downarrow x_0} f(x) = L$ iff whenever x_n is a sequence from $D \cap (x_0, \infty)$ such that $x_n \to x_0$, then $f(x_n) \to L$.*

(b) *Let x_0 be a limit point of $(-\infty, x_0) \cap D$. $\lim_{x\uparrow x_0} f(x) = L$ iff whenever x_n is a sequence from $(-\infty, x_0) \cap D$ such that $x_n \to x_0$, then $f(x_n) \to L$.*

The proof of Theorem 6.6 is similar to that of Theorem 6.2 and is left to the reader.

Theorem 6.7. *Let $f : D \to \mathbb{R}$ and x_0 be a limit point of D.*

$$\lim_{x\to x_0} f(x) = L \quad \iff \quad \lim_{x\uparrow x_0} f(x) = L = \lim_{x\downarrow x_0} f(x)$$

Proof. This proof is left as an exercise. □

Theorem 6.8. *If $f : (a, b) \to \mathbb{R}$ is monotone, then both unilateral limits of f exist at every point of (a, b).*

Proof. To be specific, suppose f is increasing and $x_0 \in (a, b)$. Let $\varepsilon > 0$ and $L = \operatorname{lub}\{f(x) : a < x < x_0\}$. According to Corollary 2.21, there must exist an $x \in (a, x_0)$ such that $L - \varepsilon < f(x) \le L$. Define $\delta = x_0 - x$. If $y \in (x_0 - \delta, x_0)$, then $L - \varepsilon < f(x) \le f(y) \le L$. This shows $\lim_{x\uparrow x_0} f(x) = L$.

The proof that $\lim_{x\downarrow x_0} f(x)$ exists is similar.

To handle the case when f is decreasing, consider $-f$ instead of f. □

6.3 Continuity

Definition 6.9. Let $f : D \to \mathbb{R}$ and $x_0 \in D$. f is *continuous at* x_0 if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that when $x \in D$ with $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$. The set of all points at which f is continuous is denoted $C(f)$.

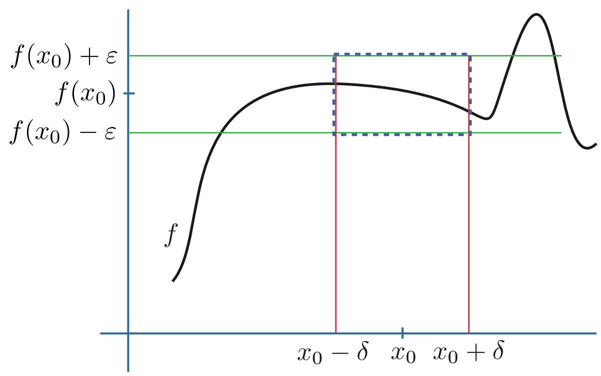


Figure 6.6: The function f is continuous at x_0, if given any $\varepsilon > 0$ there is a $\delta > 0$ such that the graph of f does not leave the dashed rectangle $(x_0 - \delta, x_0 + d) \times (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. The difference between this figure and Figure 6.1 is that in the former the point $(x_0, f(x_0))$ need not be in the rectangle, or even on the graph of f.

Several useful ways of rephrasing this are contained in the following theorem. They are analogous to the similar statements made about limits. Proofs are left to the reader.

Theorem 6.10. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. The following statements are equivalent.*

(a) $x_0 \in C(f)$,

(b) *For all $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$x \in (x_0 - \delta, x_0 + \delta) \cap D \Rightarrow f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon),$$

(c) *For all $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$f((x_0 - \delta, x_0 + \delta) \cap D) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon).$$

Example 6.10. Define

$$f(x) = \begin{cases} \dfrac{2x^2 - 8}{x - 2}, & x \neq 2 \\ 8, & x = 2 \end{cases}.$$

It follows from Example 6.3 that $2 \in C(f)$.

There is a subtle difference between the treatment of the domain of the function in the definitions of limit and continuity. In the definition of limit, the "target point," x_0 is required to be a limit point of the domain, but not actually be an element of the domain. In the definition of continuity, x_0 must be in the domain of the function, but does not have to be a limit point. To see a consequence of this difference, consider the following example.

Example 6.11. If $f : \mathbb{Z} \to \mathbb{R}$ is an arbitrary function, then $C(f) = \mathbb{Z}$. To see this, let $n_0 \in \mathbb{Z}$, $\varepsilon > 0$ and $\delta = 1$. If $x \in \mathbb{Z}$ with $|x - n_0| < \delta$, then $x = n_0$. It follows that $|f(x) - f(n_0)| = 0 < \varepsilon$, so f is continuous at n_0.

This leads to the following theorem.

Theorem 6.11. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. If x_0 is a limit point of D, then $x_0 \in C(f)$ iff $\lim_{x\to x_0} f(x) = f(x_0)$. If x_0 is an isolated point of D, then $x_0 \in C(f)$.*

Proof. If x_0 is isolated in D, then there is an $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \cap D = \{x_0\}$. For any $\varepsilon > 0$, the definition of continuity is satisfied with this δ.

Next, suppose $x_0 \in D'$.

The definition of continuity says that f is continuous at x_0 iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

The definition of limit says that $\lim_{x\to x_0} f(x) = f(x_0)$ iff for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Comparing these two definitions, it is clear that $x_0 \in C(f)$ implies

$$\lim_{x\to x_0} f(x) = f(x_0).$$

On the other hand, suppose $\lim_{x\to x_0} f(x) = f(x_0)$ and $\varepsilon > 0$. Choose δ according to the definition of limit. When $x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\}$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. It follows from this that when $x = x_0$, then $f(x) - f(x_0) = f(x_0) - f(x_0) = 0 < \varepsilon$. Therefore, when $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, and $x_0 \in C(f)$, as desired. $\square$

Example 6.12. If $f(x) = c$, for some $c \in \mathbb{R}$, then Example 6.1 and Theorem 6.11 show that f is continuous at every point.

Example 6.13. If $f(x) = x$, then Example 6.2 and Theorem 6.11 show that f is continuous at every point.

Corollary 6.12. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. $x_0 \in C(f)$ iff whenever x_n is a sequence from D with $x_n \to x_0$, then $f(x_n) \to f(x_0)$.*

Proof. Combining Theorem 6.11 with Theorem 6.2 shows this to be true. $\square$

Example 6.14 (Dirichlet Function). Suppose

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

For each $x \in \mathbb{Q}$, there is a sequence of irrational numbers converging to x, and for each $y \in \mathbb{Q}^c$ there is a sequence of rational numbers converging to y. Corollary 6.12 shows $C(f) = \emptyset$.

Example 6.15 (Salt and Pepper Function). Since $\mathbb{Q}$ is a countable set, it can be written as a sequence, $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$. Define

$$f(x) = \begin{cases} 1/n, & x = q_n, \\ 0, & x \in \mathbb{Q}^c. \end{cases}$$

If $x \in \mathbb{Q}$, then $x = q_n$, for some n and $f(x) = 1/n > 0$. There is a sequence x_n from $\mathbb{Q}^c$ such that $x_n \to x$ and $f(x_n) = 0 \not\to f(x) = 1/n$. Therefore $C(f) \cap \mathbb{Q} = \emptyset$.

On the other hand, let $x \in \mathbb{Q}^c$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ large enough so that $1/N < \varepsilon$ and let $\delta = \min\{|x - q_n| : 1 \le n \le N\}$. If $|x - y| < \delta$, there are two cases to consider. If $y \in \mathbb{Q}^c$, then $|f(y) - f(x)| = |0 - 0| = 0 < \varepsilon$. If $y \in \mathbb{Q}$, then the choice of δ guarantees $y = q_n$ for some $n > N$. In this case, $|f(y) - f(x)| = f(y) = f(q_n) = 1/n < 1/N < \varepsilon$. Therefore, $x \in C(f)$.

This shows that $C(f) = \mathbb{Q}^c$.

It is a consequence of the Baire category theorem that there is no function f such that $C(f) = \mathbb{Q}$. Proving this would take us too far afield.

The following theorem is an almost immediate consequence of Theorem 6.4.

Theorem 6.13. *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. If $x_0 \in C(f) \cap C(g)$, then*

(a) $x_0 \in C(f + g)$,

(b) $x_0 \in C(\alpha f)$, $\forall \alpha \in \mathbb{R}$,

(c) $x_0 \in C(fg)$, *and*

(d) $x_0 \in C(f/g)$ *when* $g(x_0) \neq 0$.

Corollary 6.14. *If f is a rational function, then f is continuous at each point of its domain.*

Proof. This is a consequence of Examples 6.12 and 6.13 used with Theorem 6.13. ☐

Theorem 6.15. *Suppose $f : D_f \to \mathbb{R}$ and $g : D_g \to \mathbb{R}$ such that $f(D_f) \subset D_g$. If there is an $x_0 \in C(f)$ such that $f(x_0) \in C(g)$, then $x_0 \in C(g \circ f)$.*

Proof. Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that

$$g((f(x_0) - \delta_1, f(x_0) + \delta_1) \cap D_g) \subset (g \circ f(x_0) - \varepsilon, g \circ f(x_0) + \varepsilon).$$

Choose $\delta_2 > 0$ such that

$$f((x_0 - \delta_2, x_0 + \delta_2) \cap D_f) \subset (f(x_0) - \delta_1, f(x_0) + \delta_1).$$

Then

$$\begin{aligned} g \circ f((x_0 - \delta_2, x_0 + \delta_2) \cap D_f) &\subset g((f(x_0) - \delta_1, f(x_0) + \delta_1) \cap D_g) \\ &\subset (g \circ f(x_0) - \varepsilon, g \circ f(x_0) + \varepsilon). \end{aligned}$$

Since this shows Theorem 6.10(c) is satisfied at x_0 with the function $g \circ f$, it follows that $x_0 \in C(g \circ f)$. ☐

Example 6.16. If $f(x) = \sqrt{x}$ for $x \geq 0$, then according to Problem 8, $C(f) = [0, \infty)$. Theorem 6.15 shows $f \circ f(x) = \sqrt[4]{x}$ is continuous on $[0, \infty)$.

In similar way, it can be shown by induction that $f(x) = x^{m/2^n}$ is continuous on $[0, \infty)$ for all $m, n \in \mathbb{Z}$.

6.4 Unilateral Continuity

Definition 6.16. Let $f : D \to \mathbb{R}$ and $x_0 \in D$. f is *left-continuous* at x_0 if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $f((x_0 - \delta, x_0] \cap D) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Let $f : D \to \mathbb{R}$ and $x_0 \in D$. f is *right-continuous* at x_0 if for every $\varepsilon > 0$ there is a $\delta > 0$ such that $f([x_0, x_0 + \delta) \cap D) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$.

Example 6.17. Let the *floor function* be

$$\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$$

and the *ceiling function* be

$$\lceil x \rceil = \min\{n \in \mathbb{Z} : n \geq x\}.$$

The floor function is right-continuous, but not left-continuous at each integer, and the ceiling function is left-continuous, but not right-continuous at each integer.

Theorem 6.17. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. $x_0 \in C(f)$ iff f is both right and left-continuous at x_0.*

Proof. The proof of this theorem is left as an exercise. □

According to Theorem 6.7, when f is monotone on an interval (a, b), the unilateral limits of f exist at every point. In order for such a function to be continuous at $x_0 \in (a, b)$, it must be the case that

$$\lim_{x \uparrow x_0} f(x) = f(x_0) = \lim_{x \downarrow x_0} f(x).$$

If either of the two equalities is violated, the function is not continuous at x_0.

In the case, when $\lim_{x \uparrow x_0} f(x) \neq \lim_{x \downarrow x_0} f(x)$, it is said that a *jump discontinuity* occurs at x_0.

Example 6.18. The function

$$f(x) = \begin{cases} |x|/x, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

has a jump discontinuity at $x = 0$.

In the case when $\lim_{x \uparrow x_0} f(x) = \lim_{x \downarrow x_0} f(x) \neq f(x_0)$, it is said that f has a *removable discontinuity* at x_0. The discontinuity is called "removable" because in this case, the function can be made continuous at x_0 by merely redefining its value at the single point, x_0, to be the value of the two one-sided limits.

Example 6.19. The function $f(x) = \frac{x^2-4}{x-2}$ is not continuous at $x = 2$ because 2 is not in the domain of f. Since $\lim_{x \to 2} f(x) = 4$, if the domain of f is extended to include 2 by setting $f(2) = 4$, then this extended f is continuous everywhere. (See Figure 6.7.)

Theorem 6.18. *If $f : (a, b) \to \mathbb{R}$ is monotone, then $(a, b) \setminus C(f)$ is countable.*

Proof. In light of the discussion above and Theorem 6.7, it is apparent that the only types of discontinuities f can have are jump discontinuities.

To be specific, suppose f is increasing and $x_0, y_0 \in (a, b) \setminus C(f)$ with $x_0 < y_0$. In this case, the fact that f is increasing implies

$$\lim_{x \uparrow x_0} f(x) < \lim_{x \downarrow x_0} f(x) \leq \lim_{x \uparrow y_0} f(x) < \lim_{x \downarrow y_0} f(x).$$

This implies that for any two $x_0, y_0 \in (a, b) \setminus C(f)$, there are disjoint open intervals, $I_{x_0} = (\lim_{x \uparrow x_0} f(x), \lim_{x \downarrow x_0} f(x))$ and $I_{y_0} = (\lim_{x \uparrow y_0} f(x), \lim_{x \downarrow y_0} f(x))$.

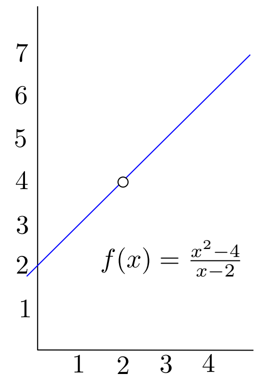


Figure 6.7: The function from Example 6.19. Note that the graph is a line with one "hole" in it. Plugging up the hole removes the discontinuity.

For each $x \in (a,b) \setminus C(f)$, choose $q_x \in I_x \cap \mathbb{Q}$. Because of the pairwise disjointness of the intervals $\{I_x : x \in (a,b) \setminus C(f)\}$, this defines an bijection between $(a,b) \setminus C(f)$ and a subset of $\mathbb{Q}$. Therefore, $(a,b) \setminus C(f)$ must be countable.

A similar argument holds for a decreasing function. $\square$

Theorem 6.18 implies that a monotone function is continuous at "nearly every" point in its domain. Characterizing the points of discontinuity as countable is the best that can be hoped for, as seen in the following example.

Example 6.20. Let $D = \{d_n : n \in \mathbb{N}\}$ be a countable set and define $J_x = \{n : d_n < x\}$. The function

$$f(x) = \begin{cases} 0, & J_x = \emptyset \\ \sum_{n \in J_x} \frac{1}{2^n} & J_x \neq \emptyset \end{cases}. \tag{6.1}$$

is increasing and $C(f) = D^c$. The proof of this statement is left as Exercise 9.

6.5 Continuous Functions

Up until now, continuity has been considered as a property of a function at a point. There is much that can be said about functions continuous everywhere.

Definition 6.19. Let $f : D \to \mathbb{R}$ and $A \subset D$. We say f *is continuous on* A if $A \subset C(f)$. If $D = C(f)$, then f is *continuous*.

Continuity at a point is, in a sense, a metric property of a function because it measures relative distances between points in the domain and image sets.

Continuity on a set becomes more of a topological property, as shown by the next few theorems.

Theorem 6.20. *$f : D \to \mathbb{R}$ is continuous iff whenever G is open in $\mathbb{R}$, then $f^{-1}(G)$ is relatively open in D.*

Proof. ($\Rightarrow$) Assume f is continuous on D and let G be open in $\mathbb{R}$. Let $x \in f^{-1}(G)$ and choose $\varepsilon > 0$ such that $(f(x) - \varepsilon, f(x) + \varepsilon) \subset G$. Using the continuity of f at x, we can find a $\delta > 0$ such that $f((x - \delta, x + \delta) \cap D) \subset G$. This implies that $(x - \delta, x + \delta) \cap D \subset f^{-1}(G)$. Because x was an arbitrary element of $f^{-1}(G)$, it follows that $f^{-1}(G)$ is open.

($\Leftarrow$) Choose $x \in D$ and let $\varepsilon > 0$. By assumption, the set $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$ is relatively open in D. This implies the existence of a $\delta > 0$ such that $(x - \delta, x + \delta) \cap D \subset f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon)$. It follows from this that $f((x - \delta, x + \delta) \cap D) \subset (f(x) - \varepsilon, f(x) + \varepsilon)$, and $x \in C(f)$. □

A function as simple as any constant function demonstrates that $f(G)$ need not be open when G is open. Defining $f : [0, \infty) \to \mathbb{R}$ by $f(x) = \sin x \tan^{-1} x$ shows that the image of a closed set need not be closed because $f([0, \infty)) = (-\pi/2, \pi/2)$.

Theorem 6.21. *If f is continuous on a compact set K, then $f(K)$ is compact.*

Proof. Let $\mathcal{O}$ be an open cover of $f(K)$ and $\mathcal{I} = \{f^{-1}(G) : G \in \mathcal{O}\}$. By Theorem 6.20, $\mathcal{I}$ is a collection of sets which are relatively open in K. Since $\mathcal{I}$ covers K, $\mathcal{I}$ is an open cover of K. Using the fact that K is compact, we can choose a finite subcover of K from $\mathcal{I}$, say $\{G_1, G_2, \ldots, G_n\}$. There are $\{H_1, H_2, \ldots, H_n\} \subset \mathcal{O}$ such that $f^{-1}(H_k) = G_k$ for $1 \le k \le n$. Then

$$f(K) \subset f\left(\bigcup_{1 \le k \le n} G_k\right) = \bigcup_{1 \le k \le n} H_k.$$

Thus, $\{H_1, H_2, \ldots, H_n\}$ is a subcover of $f(K)$ from $\mathcal{O}$. □

Several of the standard calculus theorems giving properties of continuous functions are consequences of Corollary 6.21. In a calculus course, K is usually a compact interval, $[a, b]$.

Corollary 6.22. *If $f : K \to \mathbb{R}$ is continuous and K is compact, then f is bounded.*

Proof. By Theorem 6.21, $f(K)$ is compact. Now, use the Bolzano-Weierstrass theorem to conclude f is bounded. □

Corollary 6.23 (Maximum Value Theorem). *If $f : K \to \mathbb{R}$ is continuous and K is compact, then there are $m, M \in K$ such that $f(m) \leq f(x) \leq f(M)$ for all $x \in K$.*

Proof. According to Theorem 6.21 and the Bolzano-Weierstrass theorem, $f(K)$ is closed and bounded. Because of this, $\operatorname{glb} f(K) \in f(K)$ and $\operatorname{lub} f(K) \in f(K)$. It suffices to choose $m \in f^{-1}(\operatorname{glb} f(K))$ and $M \in f^{-1}(\operatorname{lub} f(K))$. □

Corollary 6.24. *If $f : K \to \mathbb{R}$ is continuous and invertible and K is compact, then $f^{-1} : f(K) \to K$ is continuous.*

Proof. Let G be open in K. According to Theorem 6.20, it suffices to show $f(G)$ is open in $f(K)$.

To do this, note that $K \setminus G$ is compact, so by Theorem 6.21, $f(K \setminus G)$ is compact, and therefore closed. Because f is injective, $f(G) = f(K) \setminus f(K \setminus G)$. This shows $f(G)$ is open in $f(K)$. □

Theorem 6.25. *If f is continuous on an interval I, then $f(I)$ is an interval.*

Proof. If $f(I)$ is not connected, there must exist two disjoint open sets, U and V, such that $f(I) \subset U \cup V$ and $f(I) \cap U \neq \emptyset \neq f(I) \cap V$. In this case, Theorem 6.20 implies $f^{-1}(U)$ and $f^{-1}(V)$ are both open. They are clearly disjoint and $f^{-1}(U) \cap I \neq \emptyset \neq f^{-1}(V) \cap I$. But, this implies $f^{-1}(U)$ and $f^{-1}(V)$ disconnect I, which is a contradiction. Therefore, $f(I)$ is connected. □

Corollary 6.26 (Intermediate Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous and α is between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = \alpha$.*

Proof. This is an easy consequence of Theorem 6.25 and Theorem 5.23. □

Definition 6.27. A function $f : D \to \mathbb{R}$ has the *Darboux property* if whenever $a, b \in D$ and γ is between $f(a)$ and $f(b)$, then there is a c between a and b such that $f(c) = \gamma$.

Calculus texts usually call the Darboux property the *intermediate value property*. Corollary 6.26 shows that a function continuous on an interval has the Darboux property. The next example shows continuity is not necessary for the Darboux property to hold.

Example 6.21. The function

$$f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not continuous, but does have the Darboux property. (See Figure 6.4.) It can be seen from Example 6.6 that $0 \notin C(f)$.

To see f has the Darboux property, choose two numbers $a < b$.

If $a > 0$ or $b < 0$, then f is continuous on $[a, b]$ and Corollary 6.26 suffices to finish the proof.

On the other hand, if $0 \in [a, b]$, then there must exist an $n \in \mathbb{Z}$ such that both $\frac{2}{(4n+1)\pi}, \frac{2}{(4n+3)\pi} \in [a, b]$. Since $f(\frac{2}{(4n+1)\pi}) = 1$, $f(\frac{2}{(4n+3)\pi}) = -1$ and f is continuous on the interval between them, we see $f([a, b]) = [-1, 1]$, which is the entire range of f. The claim now follows.

6.6 Uniform Continuity

Most of the ideas contained in this section will not be needed until we begin developing the properties of the integral in Chapter 8.

Definition 6.28. A function $f : D \to \mathbb{R}$ is *uniformly continuous* if for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $x, y \in D$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

The idea here is that in the ordinary definition of continuity, the δ in the definition depends on both ε and the x at which continuity is being tested; i.e., δ is really a function of both ε and x. With uniform continuity, δ only depends on ε; i.e., δ is only a function of ε, and the same δ works across the whole domain.

Theorem 6.29. *If $f : D \to \mathbb{R}$ is uniformly continuous, then it is continuous.*

Proof. This proof is left as Exercise 32. □

The converse is not true.

Example 6.22. Let $f(x) = 1/x$ on $D = (0, 1)$ and $\varepsilon > 0$. It's clear that f is continuous on D. Let $\delta > 0$ and choose $m, n \in \mathbb{N}$ such that $m > 1/\delta$ and $n - m > \varepsilon$. If $x = 1/m$ and $y = 1/n$, then $0 < y < x < \delta$ and $f(y) - f(x) = n - m > \varepsilon$. Therefore, f is not uniformly continuous.

Theorem 6.30. *If $f : D \to \mathbb{R}$ is continuous and D is compact, then f is uniformly continuous.*

Proof. Suppose f is not uniformly continuous. Then there is an $\varepsilon > 0$ such that for every $n \in \mathbb{N}$ there are $x_n, y_n \in D$ with $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. An application of the Bolzano-Weierstrass theorem yields a subsequence x_{n_k} of x_n such that $x_{n_k} \to x_0 \in D$.

Since f is continuous at x_0, there is a $\delta > 0$ such that whenever $x \in (x_0 - \delta, x_0 + \delta) \cap D$, then $|f(x) - f(x_0)| < \varepsilon/2$. Choose $n_k \in \mathbb{N}$ such that $1/n_k < \delta/2$ and $x_{n_k} \in (x_0 - \delta/2, x_0 + \delta/2)$. Then both $x_{n_k}, y_{n_k} \in (x_0 - \delta, x_0 + \delta)$ and

$$\begin{aligned}\varepsilon \leq |f(x_{n_k}) - f(y_{n_k})| &= |f(x_{n_k}) - f(x_0) + f(x_0) - f(y_{n_k})| \\ &\leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| < \varepsilon/2 + \varepsilon/2 = \varepsilon,\end{aligned}$$

which is a contradiction.

Therefore, f must be uniformly continuous. □

The following corollary is an immediate consequence of Theorem 6.30.

Corollary 6.31. *If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous.*

Theorem 6.32. *Let $D \subset \mathbb{R}$ and $f : D \to \mathbb{R}$. If f is uniformly continuous and x_n is a Cauchy sequence from D, then $f(x_n)$ is a Cauchy sequence..*

Proof. The proof is left as Exercise 39. □

Uniform continuity is necessary in Theorem 6.32. To see this, let $f : (0, 1) \to \mathbb{R}$ be $f(x) = 1/x$ and $x_n = 1/n$. Then x_n is a Cauchy sequence, but $f(x_n) = n$ is not. This idea is explored in Exercise 34.

It's instructive to think about the converse to Theorem 6.32. Let $f(x) = x^2$, defined on all of $\mathbb{R}$. Since f is continuous everywhere, Corollary 6.12 shows f maps Cauchy sequences to Cauchy sequences. On the other hand, in Exercise 38, it is shown that f is not uniformly continuous. Therefore, the converse to Theorem 6.32 is false. Those functions mapping Cauchy sequences to Cauchy sequences are sometimes said to be *Cauchy continuous.* The converse to Theorem 6.32 can be tweaked to get a true statement.

Theorem 6.33. *Let $f : D \to \mathbb{R}$ where D is bounded. If f is Cauchy continuous, then f is uniformly continuous.*

Proof. Suppose f is not uniformly continuous. Then there is an $\varepsilon > 0$ and sequences x_n and y_n from D such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Since D is bounded, the sequence x_n is bounded and the Bolzano-Weierstrass theorem gives a Cauchy subsequence, x_{n_k}. The new sequence

$$z_k = \begin{cases} x_{n_{(k+1)/2}} & k \text{ odd} \\ y_{n_{k/2}} & k \text{ even} \end{cases}$$

is easily shown to be a Cauchy sequence. But, $f(z_k)$ is not a Cauchy sequence, since $|f(z_k) - f(z_{k+1})| \geq \varepsilon$ for all odd k. This contradicts the fact that f is Cauchy continuous. We're forced to conclude the assumption that f is not uniformly continuous is false. □

6.7 Exercises

Exercise 6.1. Prove $\lim_{x\to -2}(x^2+3x) = -2$.

Exercise 6.2. Give examples of functions f and g such that neither function has a limit at a, but $f+g$ does. Do the same for fg.

Exercise 6.3. Let $f : D \to \mathbb{R}$ and $a \in D'$.

$$\lim_{x\to a} f(x) = L \iff \lim_{x\uparrow a} f(x) = \lim_{x\downarrow a} f(x) = L$$

Exercise 6.4. Find two functions defined on $\mathbb{R}$ such that

$$0 = \lim_{x\to 0}(f(x)+g(x)) \neq \lim_{x\to 0} f(x) + \lim_{x\to 0} g(x).$$

Exercise 6.5. If $\lim_{x\to a} f(x) = L > 0$, then there is a $\delta > 0$ such that $f(x) > 0$ when $0 < |x-a| < \delta$.

Exercise 6.6. If $\mathbb{Q} = \{q_n : n \in \mathbb{N}\}$ is an enumeration of the rational numbers and

$$f(x) = \begin{cases} 1/n, & x = q_n \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

then $\lim_{x\to a} f(x) = 0$, for all $a \in \mathbb{Q}^c$.

Exercise 6.7. Use the definition of continuity to show $f(x) = x^2$ is continuous everywhere on $\mathbb{R}$.

Exercise 6.8. Prove that $f(x) = \sqrt{x}$ is continuous on $[0,\infty)$.

Exercise 6.9. If f is defined as in (6.1), then $D = C(f)^c$.

Exercise 6.10. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then there is a countable set D such that the values of f can be altered on D in such a way that the altered function is left-continuous at every point of $\mathbb{R}$.

Exercise 6.11. Does there exist an increasing function $f : \mathbb{R} \to \mathbb{R}$ such that $C(f) = \mathbb{Q}$?

Exercise 6.12. If $f : \mathbb{R} \to \mathbb{R}$ and there is an $\alpha > 0$ such that $|f(x) - f(y)| \leq \alpha|x - y|$ for all $x, y \in \mathbb{R}$, then show that f is continuous.

Exercise 6.13. Suppose f and g are each defined on an open interval I, $a \in I$ and $a \in C(f) \cap C(g)$. If $f(a) > g(a)$, then there is an open interval J such that $f(x) > g(x)$ for all $x \in J$.

Exercise 6.14. If $f, g : (a, b) \to \mathbb{R}$ are continuous, then $G = \{x : f(x) < g(x)\}$ is open.

Exercise 6.15. If $f : \mathbb{R} \to \mathbb{R}$ and $a \in C(f)$ with $f(a) > 0$, then there is a neighborhood G of a such that $f(G) \subset (0, \infty)$.

Exercise 6.16. Let f and g be two functions which are continuous on a set $D \subset \mathbb{R}$. Prove or give a counter example: $\{x \in D : f(x) > g(x)\}$ is relatively open in D.

Exercise 6.17. If $f, g : \mathbb{R} \to \mathbb{R}$ are functions such that $f(x) = g(x)$ for all $x \in \mathbb{Q}$ and $C(f) = C(g) = \mathbb{R}$, then $f = g$.

Exercise 6.18. Let $I = [a, b]$. If $f : I \to I$ is continuous, then there is a $c \in I$ such that $f(c) = c$.

Exercise 6.19. Find an example to show the conclusion of Problem 18 fails if $I = (a, b)$.

Exercise 6.20. If f and g are both continuous on $[a, b]$, then $\{x : f(x) \leq g(x)\}$ is compact.

Exercise 6.21. If $f : [a, b] \to \mathbb{R}$ is continuous, not constant,

$$m = \text{glb}\,\{f(x) : a \leq x \leq b\} \text{ and } M = \text{lub}\,\{f(x) : a \leq x \leq b\},$$

then $f([a, b]) = [m, M]$.

Exercise 6.22. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function such that every interval has points at which f is negative and points at which f is positive. Prove that every interval has points where f is not continuous.

Exercise 6.23. Let $\mathbb{Q} = \{p_n/q_n : p_n \in \mathbb{Z} \text{ and } q_n \in \mathbb{N}\}$ where each pair p_n and q_n are relatively prime. If

$$f(x) = \begin{cases} x, & x \in \mathbb{Q}^c \\ p_n \sin\left(\frac{1}{q_n}\right), & \frac{p_n}{q_n} \in \mathbb{Q} \end{cases},$$

then determine $C(f)$.

Exercise 6.24. If $f : [a, b] \to \mathbb{R}$ has a limit at every point, then f is bounded. Is this true for $f : (a, b) \to \mathbb{R}$?

Exercise 6.25. Give an example of a bounded function $f : \mathbb{R} \to \mathbb{R}$ with a limit at no point.

Exercise 6.26. If $f : \mathbb{R} \to \mathbb{R}$ is continuous and periodic, then there are $x_m, x_M \in \mathbb{R}$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in \mathbb{R}$. (A function f is periodic, if there is a $p > 0$ such that $f(x + p) = f(x)$ for all $x \in \mathbb{R}$. The least such p is called the *period* of f.

Exercise 6.27. A set $S \subset \mathbb{R}$ is disconnected iff there is a continuous $f : S \to \mathbb{R}$ such that $f(S) = \{0, 1\}$.

Exercise 6.28. If $f : \mathbb{R} \to \mathbb{R}$ satisfies $f(x + y) = f(x) + f(y)$ for all x and y and $0 \in C(f)$, then f is continuous.

Exercise 6.29. Assume that $f : \mathbb{R} \to \mathbb{R}$ is such that $f(x + y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. If f has a limit at zero, prove that either $\lim_{x\to 0} f(x) = 1$ or $f(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$.

Exercise 6.30. If $F \subset \mathbb{R}$ is closed, then there is an $f : \mathbb{R} \to \mathbb{R}$ such that $F = C(f)^c$.

Exercise 6.31. If $F \subset \mathbb{R}$ is closed and $f : F \to \mathbb{R}$ is continuous, then there is a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ on F.

Exercise 6.32. If $f : [a, b] \to \mathbb{R}$ is uniformly continuous, then f is continuous.

Exercise 6.33. A function $f : \mathbb{R} \to \mathbb{R}$ is *periodic* with period $p > 0$, if $f(x + p) = f(x)$ for all x. If $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p and continuous on $[0, p]$, then f is uniformly continuous.

Exercise 6.34. Prove that an unbounded function on a bounded open interval cannot be uniformly continuous.

Exercise 6.35. If $f : D \to \mathbb{R}$ is uniformly continuous on a bounded set D, then f is bounded.

Exercise 6.36. Prove Theorem 6.29.

Exercise 6.37. Every polynomial of odd degree has a root.

Exercise 6.38. Show $f(x) = x^2$, with domain $\mathbb{R}$, is not uniformly continuous.

Exercise 6.39. Prove Theorem 6.32.

Exercise 6.40. If $f(x)$ is continuous at $x = 0$, then

$$\lim_{N\to\infty} \sum_{n=0}^{N} x^n f\left(\frac{n}{N}\right) = \frac{f(0)}{1-x}$$

when $|x| < 1$.

Chapter 7

Differentiation

7.1 The Derivative at a Point

Definition 7.1. Let f be a function defined on a neighborhood of x_0. f is *differentiable* at x_0, if the following limit exists:

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

Define $D(f) = \{x : f'(x) \text{ exists}\}$.

The standard notations for the derivative will be used; e.g., $f'(x)$, $\frac{df(x)}{dx}$, $Df(x)$, etc.

Letting $h = x - x_0$ it is seen an equivalent way of stating this definition is to note that if $x_0 \in D(f)$, then

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

(See Figure 7.1.)

This can be interpreted in the standard way as the limiting slope of the line segment from $(x_0, f(x_0)$ to $(x_0 + h, f(x_0 + h))$ as the endpoints points approach each other.

Example 7.1. If $f(x) = c$ for all x and some $c \in \mathbb{R}$, then

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} = \lim_{h \to 0} \frac{c - c}{h} = 0.$$

So, $f'(x) = 0$ everywhere.

[1] March 20, 2023

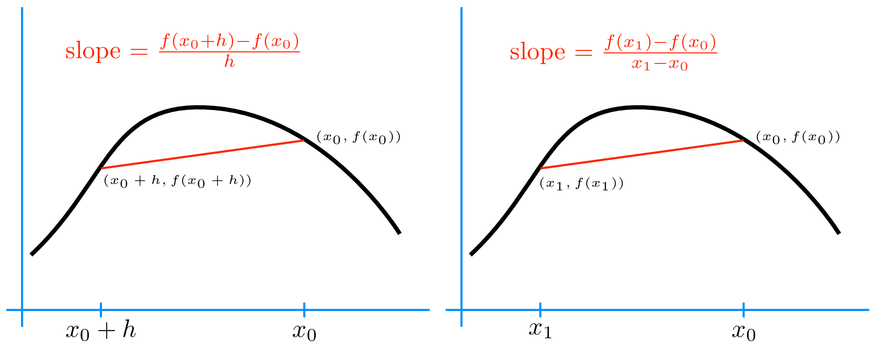


Figure 7.1: These graphs illustrate that the two standard ways of writing the difference quotient are equivalent.

Example 7.2. If $f(x) = x$, then

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0} \frac{x_0+h-x_0}{h} = \lim_{h\to 0} \frac{h}{h} = 1.$$

So, $f'(x) = 1$ everywhere.

Theorem 7.2. *For any function f, $D(f) \subset C(f)$.*

Proof. Suppose $x_0 \in D(f)$. Then

$$\begin{aligned} \lim_{x\to x_0} (f(x)-f(x_0)) &= \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}(x-x_0) \\ &= f'(x_0)\, 0 = 0. \end{aligned}$$

This shows $\lim_{x\to x_0} f(x) = f(x_0)$, and $x_0 \in C(f)$. □

Of course, the converse of Theorem 7.2 is not true.

Example 7.3. The function $f(x) = |x|$ is continuous on $\mathbb{R}$, but

$$\lim_{h\downarrow 0} \frac{f(0+h)-f(0)}{h} = 1 = -\lim_{h\uparrow 0} \frac{f(0+h)-f(0)}{h},$$

so $f'(0)$ fails to exist.

Theorem 7.2 and Example 7.3 show that differentiability is a strictly stronger condition than continuity. For a long time most mathematicians believed that every continuous function must certainly be differentiable at some point. In the nineteenth century, several researchers, most notably Bolzano and Weierstrass, presented examples of functions continuous everywhere and differentiable nowhere.[2] It has since been proved that, in a technical sense, the "typical" continuous function is nowhere differentiable [5]. So, contrary to the impression left by many beginning calculus courses, differentiability is the exception rather than the rule, even for continuous functions..

7.2 Differentiation Rules

Following are the standard rules for differentiation learned by every calculus student.

Theorem 7.3. *Suppose f and g are functions such that $x_0 \in D(f) \cap D(g)$.*

(a) $x_0 \in D(f+g)$ *and* $(f+g)'(x_0) = f'(x_0) + g'(x_0)$.

(b) *If* $a \in \mathbb{R}$, *then* $x_0 \in D(af)$ *and* $(af)'(x_0) = af'(x_0)$.

(c) $x_0 \in D(fg)$ *and* $(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$.

(d) *If* $g(x_0) \neq 0$, *then* $x_0 \in D(f/g)$ *and*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Proof. (a)

$$\begin{aligned}\lim_{h\to 0} &\frac{(f+g)(x_0+h) - (f+g)(x_0)}{h} \\ &= \lim_{h\to 0} \frac{f(x_0+h) + g(x_0+h) - f(x_0) - g(x_0)}{h} \\ &= \lim_{h\to 0}\left(\frac{f(x_0+h) - f(x_0)}{h} + \frac{g(x_0+h) - g(x_0)}{h}\right) = f'(x_0) + g'(x_0)\end{aligned}$$

(b)

$$\lim_{h\to 0} \frac{(af)(x_0+h) - (af)(x_0)}{h} = a \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} = af'(x_0)$$

[2] Bolzano presented his example in 1834, but it was little noticed. The 1872 example of Weierstrass is more well-known [3]. A translation of Weierstrass' original paper [22] is presented by Edgar [12]. Weierstrass' example is not very transparent because it depends on trigonometric series. Many more elementary constructions have since been made. One such will be presented in Example 9.9.

(c)

$$\lim_{h\to 0} \frac{(fg)(x_0+h)-(fg)(x_0)}{h} = \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}$$

Now, "slip a 0" into the numerator and factor the fraction.

$$= \lim_{h\to 0} \frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0+h)+f(x_0)g(x_0+h)-f(x_0)g(x_0)}{h}$$

$$= \lim_{h\to 0} \left(\frac{f(x_0+h)-f(x_0)}{h} g(x_0+h) + f(x_0) \frac{g(x_0+h)-g(x_0)}{h} \right)$$

Finally, use the definition of the derivative and the continuity of f and g at x_0.

$$= f'(x_0)g(x_0) + f(x_0)g'(x_0)$$

(d) It will be proved that if $g(x_0) \neq 0$, then $(1/g)'(x_0) = -g'(x_0)/(g(x_0))^2$. This statement, combined with (c), yields (d).

$$\begin{aligned}
\lim_{h\to 0} \frac{(1/g)(x_0+h)-(1/g)(x_0)}{h} &= \lim_{h\to 0} \frac{\dfrac{1}{g(x_0+h)} - \dfrac{1}{g(x_0)}}{h} \\
&= \lim_{h\to 0} \frac{g(x_0)-g(x_0+h)}{h} \frac{1}{g(x_0+h)g(x_0)} \\
&= -\frac{g'(x_0)}{(g(x_0)^2}
\end{aligned}$$

Plug this into (c) to see

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= \left(f\frac{1}{g}\right)'(x_0) \\
&= f'(x_0)\frac{1}{g(x_0)} + f(x_0)\frac{-g'(x_0)}{(g(x_0))^2} \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.
\end{aligned}$$

□

Combining Examples 7.1 and 7.2 with Theorem 7.3, the following theorem is easy to prove.

Corollary 7.4. *A rational function is differentiable at every point of its domain.*

Theorem 7.5 (Chain Rule). *If f and g are functions such that $x_0 \in D(f)$ and $f(x_0) \in D(g)$, then $x_0 \in D(g \circ f)$ and $(g \circ f)'(x_0) = g' \circ f(x_0) f'(x_0)$.*

Proof. Let $y_0 = f(x_0)$. By assumption, there is an open interval J containing $f(x_0)$ such that g is defined on J. Since J is open and $x_0 \in C(f)$, there is an open interval I containing x_0 such that $f(I) \subset J$.

Define $h : J \to \mathbb{R}$ by

$$h(y) = \begin{cases} \dfrac{g(y) - g(y_0)}{y - y_0} - g'(y_0), & y \neq y_0 \\ 0, & y = y_0 \end{cases}.$$

Since $y_0 \in D(g)$, we see

$$\lim_{y \to y_0} h(y) = \lim_{y \to y_0} \frac{g(y) - g(y_0)}{y - y_0} - g'(y_0) = g'(y_0) - g'(y_0) = 0 = h(y_0),$$

so $y_0 \in C(h)$. Now, $x_0 \in C(f)$ and $f(x_0) = y_0 \in C(h)$, so Theorem 6.15 implies $x_0 \in C(h \circ f)$. In particular

$$\lim_{x \to x_0} h \circ f(x) = 0. \tag{7.1}$$

From the definition of $h \circ f$ for $x \in I$ with $f(x) \neq f(x_0)$, we can solve for

$$g \circ f(x) - g \circ f(x_0) = (h \circ f(x) + g' \circ f(x_0))(f(x) - f(x_0)). \tag{7.2}$$

Notice that (7.2) is also true when $f(x) = f(x_0)$. Divide both sides of (7.2) by $x - x_0$, and use (7.1) to obtain

$$\begin{aligned} \lim_{x \to x_0} \frac{g \circ f(x) - g \circ f(x_0)}{x - x_0} &= \lim_{x \to x_0} (h \circ f(x) + g' \circ f(x_0)) \frac{f(x) - f(x_0)}{x - x_0} \\ &= (0 + g' \circ f(x_0)) f'(x_0) \\ &= g' \circ f(x_0) f'(x_0). \end{aligned}$$

□

Theorem 7.6. *Suppose $f : [a, b] \to [c, d]$ is continuous and invertible. If $x_0 \in D(f)$ and $f'(x_0) \neq 0$ for some $x_0 \in (a, b)$, then $f(x_0) \in D(f^{-1})$ and $\left(f^{-1}\right)' (f(x_0)) = 1/f'(x_0)$.*

Proof. Let $y_0 = f(x_0)$ and suppose y_n is any sequence in $f([a,b]) \setminus \{y_0\}$ converging to y_0 and $x_n = f^{-1}(y_n)$. By Theorem 6.24, f^{-1} is continuous, so

$$x_0 = f^{-1}(y_0) = \lim_{n\to\infty} f^{-1}(y_n) = \lim_{n\to\infty} x_n.$$

Therefore,

$$\lim_{n\to\infty} \frac{f^{-1}(y_n) - f^{-1}(y_0)}{y_n - y_0} = \lim_{n\to\infty} \frac{x_n - x_0}{f(x_n) - f(x_0)} = \frac{1}{f'(x_0)}.$$

$\square$

Example 7.4. It follows easily from Theorem 7.3 that $f(x) = x^3$ is differentiable everywhere with $f'(x) = 3x^2$. Define $g(x) = \sqrt[3]{x}$. Then $g(x) = f^{-1}(x)$. Suppose $g(y_0) = x_0$ for some $y_0 \in \mathbb{R}$. According to Theorem 7.6,

$$g'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{3x_0^2} = \frac{1}{3(g(y_0))^2} = \frac{1}{3(\sqrt[3]{y_0})^2} = \frac{1}{3y_0^{2/3}}. \tag{7.3}$$

If $h(x) = x^{2/3}$, then $h(x) = g(x)^2$, so (7.3) and the Chain Rule show

$$h'(x) = \frac{2}{3\sqrt[3]{x}},\ x \neq 0,$$

as expected.

In the same manner as Example 7.4, the usual power rule for differentiation can be proved.

Corollary 7.7. *Suppose $q \in \mathbb{Q}$, $f(x) = x^q$ and D is the domain of f. Then $f'(x) = qx^{q-1}$ on the set*

$$\begin{cases} D, & \text{when } q \geq 1 \\ D \setminus \{0\}, & \text{when } q < 1 \end{cases}.$$

7.3 Derivatives and Extreme Points

As learned in calculus, the derivative is a powerful tool for determining the behavior of functions. The following theorems form the basis for much of differential calculus. First, we state a few familiar definitions.

Definition 7.8. Suppose $f : D \to \mathbb{R}$ and $x_0 \in D$. f is said to have a *relative maximum at x_0* if there is a $\delta > 0$ such that $f(x) \leq f(x_0)$ for all $x \in (x_0 - \delta, x_0 + \delta) \cap D$. f has a *relative minimum at x_0* if $-f$ has a relative maximum at x_0. If f

has either a relative maximum or a relative minimum at x_0, then it is said that *f has a relative extreme value at x_0.*

The *absolute maximum* of f occurs at x_0 if $f(x_0) \geq f(x)$ for all $x \in D$. The definitions of *absolute minimum* and *absolute extreme* are analogous.

Examples like $f(x) = x$ on $(0,1)$ show that even the nicest functions need not have relative extrema.

Theorem 7.9. *Suppose $f : (a,b) \to \mathbb{R}$. If f has a relative extreme value at x_0 and $x_0 \in D(f)$, then $f'(x_0) = 0$.*

Proof. Suppose $f(x_0)$ is a relative maximum value of f. Then there must be a $\delta > 0$ such that $f(x) \leq f(x_0)$ whenever $x \in (x_0 - \delta, x_0 + \delta)$. Since $f'(x_0)$ exists,

$$x \in (x_0 - \delta, x_0) \implies \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \implies f'(x_0) = \lim_{x \uparrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \geq 0 \tag{7.4}$$

and

$$x \in (x_0, x_0 + \delta) \implies \frac{f(x) - f(x_0)}{x - x_0} \leq 0 \implies f'(x_0) = \lim_{x \downarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \leq 0. \tag{7.5}$$

Combining (7.4) and (7.5) shows $f'(x_0) = 0$.

If $f(x_0)$ is a relative minimum value of f, apply the previous argument to $-f$. □

Suppose $f : [a,b] \to \mathbb{R}$ is continuous. Corollary 6.23 guarantees f has both an absolute maximum and minimum on the compact interval $[a,b]$. Theorem 7.9 implies these extrema must occur at points of the set

$$C = \{x \in (a,b) : f'(x) = 0\} \cup \{x \in [a,b] : f'(x) \text{ does not exist}\}.$$

The elements of C are often called the *critical points* or *critical numbers* of f on $[a,b]$. To find the maximum and minimum values of f on $[a,b]$, it suffices to find its maximum and minimum on the smaller set C, which is often finite.

7.4 Differentiable Functions

Differentiation becomes most useful when a function has a derivative at each point of an interval.

[3] March 20, 2023

Definition 7.10. Let G be an open set. The function f is *differentiable on an G* if $G \subset D(f)$. The set of all functions differentiable on G is denoted $D(G)$. If f is differentiable on its domain, then it is said to be *differentiable*. In this case, the function f' is called the *derivative* of f.

The fundamental theorem about differentiable functions is the Mean Value Theorem. Following is its simplest form.

Lemma 7.11 (Rolle's Theorem)**.** *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$, differentiable on (a,b) and $f(a) = 0 = f(b)$, then there is a $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. Since $[a,b]$ is compact, Corollary 6.23 implies the existence of $x_m, x_M \in [a,b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a,b]$. If $f(x_m) = f(x_M)$, then f is constant on $[a,b]$ and any $c \in (a,b)$ satisfies the lemma. Otherwise, either $f(x_m) < 0$ or $f(x_M) > 0$. If $f(x_m) < 0$, then $x_m \in (a,b)$ and Theorem 7.9 implies $f'(x_m) = 0$. If $f(x_M) > 0$, then $x_M \in (a,b)$ and Theorem 7.9 implies $f'(x_M) = 0$. □

Rolle's Theorem is just a stepping-stone on the path to the Mean Value Theorem. Two versions of the Mean Value Theorem follow. The first is a version more general than the one given in most calculus courses. The second is the usual version.[4]

Theorem 7.12 (Cauchy Mean Value Theorem)**.** *If $f : [a,b] \to \mathbb{R}$ and $g : [a,b] \to \mathbb{R}$ are both continuous on $[a,b]$ and differentiable on (a,b), then there is a $c \in (a,b)$ such that*

$$g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)).$$

Proof. Let

$$h(x) = (g(b) - g(a))(f(a) - f(x)) + (g(x) - g(a))(f(b) - f(a)).$$

Because of the assumptions on f and g, h is continuous on $[a,b]$ and differentiable on (a,b) with $h(a) = h(b) = 0$. Theorem 7.11 implies there is a $c \in (a,b)$ such that $h'(c) = 0$. Then

$$\begin{aligned} 0 = h'(c) &= -(g(b) - g(a))f'(c) + g'(c)(f(b) - f(a)) \\ &\implies g'(c)(f(b) - f(a)) = f'(c)(g(b) - g(a)). \end{aligned}$$

□

[4]Theorem 7.12 is also sometimes called the *Generalized Mean Value Theorem.*

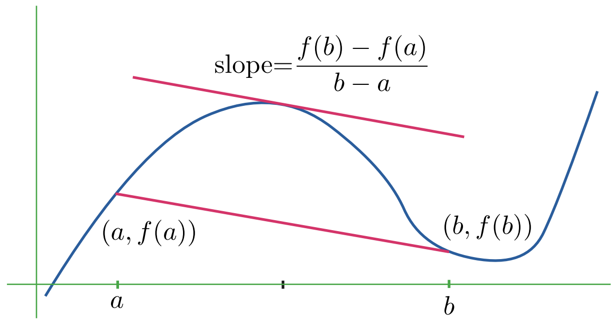


Figure 7.2: This is a "picture proof" of Corollary 7.13.

Corollary 7.13 (Mean Value Theorem). *If $f : [a,b] \to \mathbb{R}$ is continuous on $[a,b]$ and differentiable on (a,b), then there is a $c \in (a,b)$ such that $f(b) - f(a) = f'(c)(b-a)$.*

Proof. Let $g(x) = x$ in Theorem 7.12. □

Many of the standard theorems of elementary calculus are easy consequences of the Mean Value Theorem. For example, following are the usual theorems about monotonicity.

First, recall the following definitions.

Definition 7.14. A function $f : (a,b) \to \mathbb{R}$ is *increasing* on (a,b), if $a < x < y < b$ implies $f(x) \leq f(y)$. It is *decreasing*, if $-f$ is increasing. When it is increasing or decreasing, it is *monotone*.

Notice with these definitions, a constant function is both increasing and decreasing. In the case when $a < x < y < b$ implies $f(x) < f(y)$, then f is *strictly increasing*. The definition of *strictly decreasing* is analogous.

Theorem 7.15. *Suppose $f : (a,b) \to \mathbb{R}$ is a differentiable function. f is increasing on (a,b) iff $f'(x) \geq 0$ for all $x \in (a,b)$. f is decreasing on (a,b) iff $f'(x) \leq 0$ for all $x \in (a,b)$.*

Proof. Only the first assertion is proved because the proof of the second is pretty much the same with all the inequalities reversed.

($\Rightarrow$) If $x, y \in (a,b)$ with $x < y$, then the assumption that f is increasing gives

$$\frac{f(y)-f(x)}{y-x} \geq 0 \implies f'(x) = \lim_{y \to x} \frac{f(y)-f(x)}{y-x} \geq 0.$$

($\Leftarrow$) Let $x, y \in (a, b)$ with $x < y$. According to Theorem 7.13, there is a $c \in (x, y)$ such that $f(y) - f(x) = f'(c)(y - x) \geq 0$. This shows $f(x) \leq f(y)$, so f is increasing on (a, b). $\square$

Corollary 7.16. *Let $f : (a, b) \to \mathbb{R}$ be a differentiable function. f is constant iff $f'(x) = 0$ for all $x \in (a, b)$.*

It follows from Theorem 7.2 that every differentiable function is continuous. But, it's not true that a *derivative* must be continuous.

Example 7.5. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

We claim f is differentiable everywhere, but f' is not continuous.

To see this, first note that when $x \neq 0$, the standard differentiation formulas give that $f'(x) = 2x \sin(1/x) - \cos(1/x)$. To calculate $f'(0)$, choose any $h \neq 0$. Then

$$\left| \frac{f(h)}{h} \right| = \left| \frac{h^2 \sin(1/h)}{h} \right| \leq \left| \frac{h^2}{h} \right| = |h|$$

and it easily follows from the definition of the derivative and the Squeeze Theorem (Theorem 6.3) that $f'(0) = 0$.

Therefore,

$$f'(x) = \begin{cases} 0, & x = 0 \\ 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \neq 0 \end{cases}.$$

Let $x_n = 1/2\pi n$ for $n \in \mathbb{N}$. Then $x_n \to 0$ and

$$\begin{aligned} f'(x_n) &= 2x_n \sin(1/x_n) - \cos(1/x_n) \\ &= \frac{1}{\pi n} \sin 2\pi n - \cos 2\pi n = -1 \end{aligned}$$

for all n. Therefore, $f'(x_n) \to -1 \neq 0 = f'(0)$, and f' is not continuous at 0.

But, derivatives do share one useful property with continuous functions; they satisfy an intermediate value property. Compare the following theorem with Corollary 6.26.

Theorem 7.17 (Darboux's Theorem)**.** *If f is differentiable on an open set containing $[a, b]$ and γ is between $f'(a)$ and $f'(b)$, then there is a $c \in [a, b]$ such that $f'(c) = \gamma$.*

Proof. If $f'(a) = f'(b)$, then $c = a$ satisfies the theorem. So, we may as well assume $f'(a) \neq f'(b)$. There is no generality lost in assuming $f'(a) < f'(b)$, for, otherwise, we just replace f with $g = -f$.

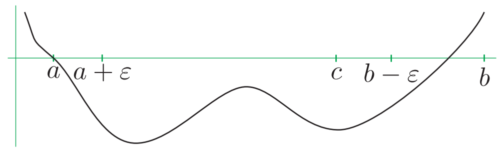


Figure 7.3: This could be the function h of Theorem 7.17.

Let $h(x) = f(x) - \gamma x$ so that $D(f) = D(h)$ and $h'(x) = f'(x) - \gamma$. In particular, this implies $h'(a) < 0 < h'(b)$. Because of this, there must be an $\varepsilon > 0$ small enough so that

$$\frac{h(a+\varepsilon) - h(a)}{\varepsilon} < 0 \implies h(a+\varepsilon) < h(a)$$

and

$$\frac{h(b) - h(b-\varepsilon)}{\varepsilon} > 0 \implies h(b-\varepsilon) < h(b).$$

(See Figure 7.3.) In light of these two inequalities and Theorem 6.23, there must be a $c \in (a,b)$ such that $h(c) = \text{glb}\,\{h(x) : x \in [a,b]\}$. Now Theorem 7.9 gives $0 = h'(c) = f'(c) - \gamma$, and the theorem follows. □

Here's an example showing a possible use of Theorem 7.17.

Example 7.6. Let

$$f(x) = \begin{cases} 0, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Theorem 7.17 implies f is not a derivative.

A more striking example is the following

Example 7.7. Define

$$f(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases} \text{ and } g(x) = \begin{cases} \sin\frac{1}{x}, & x \neq 0 \\ -1, & x = 0 \end{cases}.$$

Since

$$f(x) - g(x) = \begin{cases} 0, & x \neq 0 \\ 2, & x = 0 \end{cases}$$

does not have the intermediate value property, at least one of f or g is not a derivative. (Actually, neither is a derivative because $f(x) = -g(-x)$.)

7.5 Applications of the Mean Value Theorem

In the following sections, the standard notion of higher order derivatives is used. To make this precise, suppose f is defined on an interval I. The function f itself can be written $f^{(0)}$. If f is differentiable, then f' is written $f^{(1)}$. Continuing inductively, if $n \in \omega$, $f^{(n)}$ exists on I and $x_0 \in D(f^{(n)})$, then $f^{(n+1)}(x_0) = df^{(n)}(x_0)/dx$.

7.5.1 Taylor's Theorem

The motivation behind Taylor's theorem is the attempt to approximate a function f near a number a by a polynomial. The polynomial of degree 0 which does the best job is clearly $p_0(x) = f(a)$. The best polynomial of degree 1 is the tangent line to the graph of the function $p_1(x) = f(a) + f'(a)(x-a)$. Continuing in this way, we approximate f near a by the polynomial p_n of degree n such that $f^{(k)}(a) = p_n^{(k)}(a)$ for $k = 0, 1, \ldots, n$. A simple induction argument shows that

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k. \tag{7.6}$$

This is the well-known Taylor polynomial for f at a.

Many students leave calculus with the mistaken impression that (7.6) is the important part of Taylor's theorem. But, the important part of Taylor's theorem is the fact that in many cases it is possible to determine how large n must be to achieve a desired accuracy in the approximation of f; i. e., the error term is the important part.

Theorem 7.18 (Taylor's Theorem)**.** *If f is a function such that $f, f', \ldots, f^{(n)}$ are continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b), then there is a $c \in (a,b)$ such that*

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Proof. Let the constant α be defined by

$$f(b) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(b-a)^k + \frac{\alpha}{(n+1)!}(b-a)^{n+1} \tag{7.7}$$

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and define

$$F(x) = f(b) - \left(\sum_{k=0}^{n} \frac{f^{(k)}(x)}{k!}(b-x)^k + \frac{\alpha}{(n+1)!}(b-x)^{n+1} \right).$$

From (7.7) we see that $F(a) = 0$. Direct substitution in the definition of F shows that $F(b) = 0$. From the assumptions in the statement of the theorem, it is easy to see that F is continuous on $[a, b]$ and differentiable on (a, b). An application of Rolle's Theorem yields a $c \in (a, b)$ such that

$$0 = F'(c) = -\left(\frac{f^{(n+1)}(c)}{n!}(b-c)^n - \frac{\alpha}{n!}(b-c)^n \right) \implies \alpha = f^{(n+1)}(c),$$

as desired. □

Now, suppose f is defined on an open interval I with $a, x \in I$. If f is $n+1$ times differentiable on I, then Theorem 7.18 implies there is a c between a and x such that

$$f(x) = p_n(x) + R_f(n, x, a),$$

where $R_f(n, x, a) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$ is the error in the approximation.[6]

Example 7.8. Let $f(x) = \cos x$. Suppose we want to approximate $f(2)$ to 5 decimal places of accuracy. Since it's an easy point to work with, we'll choose $a = 0$. Then, for some $c \in (0, 2)$,

$$|R_f(n, 2, 0)| = \frac{|f^{(n+1)}(c)|}{(n+1)!}2^{n+1} \le \frac{2^{n+1}}{(n+1)!}. \tag{7.8}$$

A bit of experimentation with a calculator shows that $n = 12$ is the smallest n such that the right-hand side of (7.8) is less than 5×10^{-6}. After doing some arithmetic, it follows that

$$p_{12}(2) = 1 - \frac{2^2}{2!} + \frac{2^4}{4!} - \frac{2^6}{6!} + \frac{2^8}{8!} - \frac{2^{10}}{10!} + \frac{2^{12}}{12!} = -\frac{27809}{66825} \approx -0.41615.$$

is a 5 decimal place approximation to $\cos(2)$. (A calculator gives the value $\cos(2) = -0.416146836547142$ which is about 0.00000316 larger, comfortably less than the desired maximum error.)

[6]There are several different formulas for the error. The one given here is sometimes called the *Lagrange form of the remainder*. In Example 8.4 a form of the remainder using integration instead of differentiation is derived.

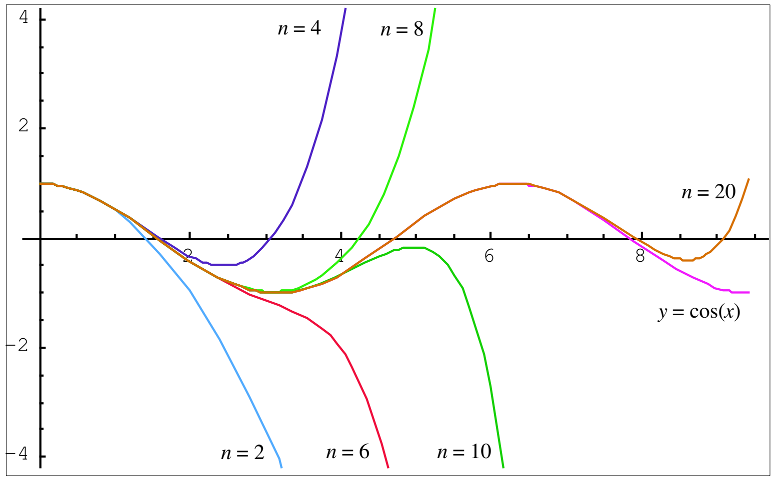

Figure 7.4: Here are several of the Taylor polynomials for the function $\cos(x)$, centered at $a = 0$, graphed along with $\cos(x)$.

But, things don't always work out the way we might like. Consider the following example.

Example 7.9. Suppose

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Figure 7.5 below has a graph of this function. In Example 7.11 below it is shown that f is differentiable to all orders everywhere and $f^{(n)}(0) = 0$ for all $n \geq 0$. With this function the Taylor polynomial centered at 0 gives a useless approximation.

7.5.2 L'Hôpital's Rules and Indeterminate Forms

According to Theorem 6.4, when $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ both exist, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$$

as long as $\lim_{x \to a} g(x) \neq 0$. But, it is easy to find examples where both $\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ and $\lim_{x \to a} f(x)/g(x)$ exists, as well as similar examples where $\lim_{x \to a} f(x)/g(x)$ fails to exist. Because of this, such a limit problem is said to be in the *indeterminate form* $0/0$. The following theorem allows us to determine many such limits.

Theorem 7.19 (Easy L'Hôpital's Rule)**.** *Suppose f and g are each continuous on $[a,b]$, differentiable on (a,b) and $f(b) = g(b) = 0$. If $g'(x) \neq 0$ on (a,b) and $\lim_{x\uparrow b} f'(x)/g'(x) = L$, where L could be infinite, then $\lim_{x\uparrow b} f(x)/g(x) = L$.*

Proof. Let $x \in [a,b)$, so f and g are continuous on $[x,b]$ and differentiable on (x,b). Cauchy's Mean Value Theorem, Theorem 7.12, implies there is a $c(x) \in (x,b)$ such that

$$f'(c(x))g(x) = g'(c(x))f(x) \implies \frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))}.$$

Since $x < c(x) < b$, it follows that $\lim_{x\uparrow b} c(x) = b$. This shows that

$$L = \lim_{x\uparrow b} \frac{f'(x)}{g'(x)} = \lim_{x\uparrow b} \frac{f'(c(x))}{g'(c(x))} = \lim_{x\uparrow b} \frac{f(x)}{g(x)}.$$

$\square$

Several things should be noted about this theorem.

First, there is nothing special about the left-hand limit used in the statement of the theorem. It could just as easily be written in terms of the right-hand limit.

Second, if $\lim_{x\to a} f(x)/g(x)$ is not of the indeterminate form $0/0$, then applying L'Hôpital's rule will usually give a wrong answer. To see this, consider

$$\lim_{x\to 0} \frac{x}{x+1} = 0 \neq 1 = \lim_{x\to 0} \frac{1}{1}.$$

Third, be careful with the flow of logic in the theorem! Consider

$$\lim_{x\to 0} \frac{x^2 \sin(1/x)}{\sin x} = \lim_{x\to 0} \frac{x}{\sin x} x \sin(1/x) = 1 \times 0 = 0.$$

On the other hand, even though we have a $0/0$ indeterminate form, blindly trying to apply L'Hôpital's Rule gives

$$\lim_{x\to 0} \frac{2x\sin(1/x) - \cos(1/x)}{\cos x},$$

which does not exist.

Another case where the indeterminate form $0/0$ occurs is in the limit at infinity. That L'Hôpital's rule works in this case can easily be deduced from Theorem 7.19.

Corollary 7.20. *Suppose f and g are differentiable on (a,∞) and*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0.$$

If $g'(x) \neq 0$ on (a,∞) and $\lim_{x\to\infty} f'(x)/g'(x) = L$, where L could be infinite, then $\lim_{x\to\infty} f(x)/g(x) = L$.

Proof. There is no generality lost by assuming $a > 0$. Let

$$F(x) = \begin{cases} f(1/x), & x \in (0,1/a] \\ 0, & x = 0 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} g(1/x), & x \in (0,1/a] \\ 0, & x = 0 \end{cases}.$$

Then

$$\lim_{x\downarrow 0} F(x) = \lim_{x\to\infty} f(x) = 0 = \lim_{x\to\infty} g(x) = \lim_{x\downarrow 0} G(x),$$

so both F and G are continuous at 0. It follows that both F and G are continuous on $[0,1/a]$ and differentiable on $(0,1/a)$ with $G'(x) = -g'(1/x)/x^2 \neq 0$ on $(0,1/a)$ and $\lim_{x\downarrow 0} F'(x)/G'(x) = \lim_{x\to\infty} f'(x)/g'(x) = L$. The rest follows from Theorem 7.19. □

The other standard indeterminate form arises when

$$\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x).$$

This is called an ∞/∞ indeterminate form. It is often handled by the following theorem.

Theorem 7.21 (Hard L'Hôpital's Rule)**.** *Suppose that f and g are differentiable on (a,∞) and $g'(x) \neq 0$ on (a,∞). If*

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty \quad \textit{and} \quad \lim_{x\to\infty} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{-\infty,\infty\},$$

then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L.$$

Proof. First, suppose $L \in \mathbb{R}$ and let $\varepsilon > 0$. Choose $a_1 > a$ large enough so that

$$\left| \frac{f'(x)}{g'(x)} - L \right| < \varepsilon, \ \forall x > a_1.$$

Since $\lim_{x\to\infty} f(x) = \infty = \lim_{x\to\infty} g(x)$, we can assume there is an $a_2 > a_1$ such that both $f(x) > 0$ and $g(x) > 0$ when $x > a_2$. Finally, choose $a_3 > a_2$ such that whenever $x > a_3$, then $f(x) > f(a_2)$ and $g(x) > g(a_2)$.

Let $x > a_3$ and apply Cauchy's Mean Value Theorem, Theorem 7.12, to f and g on $[a_2, x]$ to find a $c(x) \in (a_2, x)$ such that

$$\frac{f'(c(x))}{g'(c(x))} = \frac{f(x) - f(a_2)}{g(x) - g(a_2)} = \frac{f(x)\left(1 - \frac{f(a_2)}{f(x)}\right)}{g(x)\left(1 - \frac{g(a_2)}{g(x)}\right)}. \tag{7.9}$$

If

$$h(x) = \frac{1 - \frac{g(a_2)}{g(x)}}{1 - \frac{f(a_2)}{f(x)}},$$

then (7.9) implies

$$\frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))} h(x).$$

Since $\lim_{x \to \infty} h(x) = 1$, there is an $a_4 > a_3$ such that whenever $x > a_4$, then $|h(x) - 1| < \varepsilon$. If $x > a_4$, then

$$\begin{aligned}
\left| \frac{f(x)}{g(x)} - L \right| &= \left| \frac{f'(c(x))}{g'(c(x))} h(x) - L \right| \\
&= \left| \frac{f'(c(x))}{g'(c(x))} h(x) - Lh(x) + Lh(x) - L \right| \\
&\le \left| \frac{f'(c(x))}{g'(c(x))} - L \right| |h(x)| + |L||h(x) - 1| \\
&< \varepsilon(1 + \varepsilon) + |L|\varepsilon = (1 + |L| + \varepsilon)\varepsilon
\end{aligned}$$

can be made arbitrarily small through a proper choice of ε. Therefore

$$\lim_{x \to \infty} f(x)/g(x) = L.$$

The case when $L = \infty$ is done similarly by first choosing a $B > 0$ and adjusting (7.9) so that $f'(x)/g'(x) > B$ when $x > a_1$. A similar adjustment is necessary when $L = -\infty$. □

There is a companion corollary to Theorem 7.21 which is proved in the same way as Corollary 7.20.

Corollary 7.22. *Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b) with $g'(x) \neq 0$ on (a, b). If*

$$\lim_{x \downarrow a} f(x) = \lim_{x \downarrow a} g(x) = \infty \quad \text{and} \quad \lim_{x \downarrow a} \frac{f'(x)}{g'(x)} = L \in \mathbb{R} \cup \{-\infty, \infty\},$$

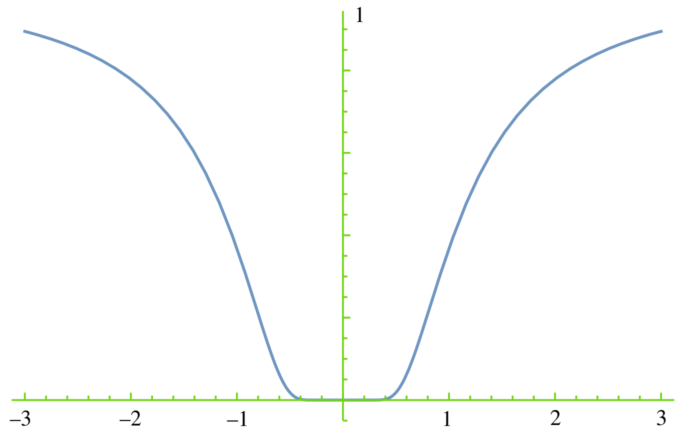

Figure 7.5: This is a plot of $f(x) = \exp(-1/x^2)$. Notice how the graph flattens out near the origin.

then

$$\lim_{x \downarrow a} \frac{f(x)}{g(x)} = L.$$

Example 7.10. If $\alpha > 0$, then $\lim_{x\to\infty} \ln x / x^\alpha$ is of the indeterminate form ∞/∞. Taking derivatives of the numerator and denominator yields

$$\lim_{x\to\infty} \frac{1/x}{\alpha x^{\alpha-1}} = \lim_{x\to\infty} \frac{1}{\alpha x^\alpha} = 0.$$

Theorem 7.21 now implies $\lim_{x\to\infty} \ln x / x^\alpha = 0$, and therefore $\ln x$ increases more slowly than any positive power of x.

Example 7.11. Let f be as in Example 7.9. (See Figure 7.5.) It is clear $f^{(n)}(x)$ exists whenever $n \in \omega$ and $x \neq 0$. We claim $f^{(n)}(0) = 0$. To see this, we first prove that

$$\lim_{x\to 0} \frac{e^{-1/x^2}}{x^n} = 0, \ \forall n \in \mathbb{Z}. \tag{7.10}$$

When $n \leq 0$, (7.10) is obvious. So, suppose (7.10) is true whenever $m \leq n$ for some $n \in \omega$. Making the substitution $u = 1/x$, we see

$$\lim_{x \downarrow 0} \frac{e^{-1/x^2}}{x^{n+1}} = \lim_{u\to\infty} \frac{u^{n+1}}{e^{u^2}}. \tag{7.11}$$

The right-hand side is an ∞/∞ indeterminate form, so L'Hôpital's rule can be used. Since

$$\lim_{u\to\infty} \frac{(n+1)u^n}{2ue^{u^2}} = \lim_{u\to\infty} \frac{(n+1)u^{n-1}}{2e^{u^2}} = \frac{n+1}{2} \lim_{x\downarrow 0} \frac{e^{-1/x^2}}{x^{n-1}} = 0$$

by the inductive hypothesis, Theorem 7.21 gives (7.11) in the case of the right-hand limit. The left-hand limit is handled similarly. Finally, (7.10) follows by induction.

When $x \neq 0$, a bit of experimentation can convince the reader that $f^{(n)}(x)$ is of the form $p_n(1/x)e^{-1/x^2}$, where p_n is a polynomial. Induction and repeated applications of (7.10) establish that $f^{(n)}(0) = 0$ for $n \in \omega$.

7.6 Exercises

Exercise 7.1. If

$$f(x) = \begin{cases} x^2, & x \in \mathbb{Q} \\ 0, & \text{otherwise} \end{cases},$$

then show $D(f) = \{0\}$ and find $f'(0)$.

Exercise 7.2. Let f be a function defined on some neighborhood of a with $f(a) = 0$. Prove $f'(a) = 0$ if and only if $a \in D(|f|)$.

Exercise 7.3. If f is defined on an open set containing x_0, the *symmetric derivative* of f at x_0 is defined as

$$f^s(x_0) = \lim_{h\to 0} \frac{f(x_0+h) - f(x_0-h)}{2h}.$$

Prove that if $f'(x)$ exists, then so does $f^s(x)$. Is the converse true?

Exercise 7.4. Let G be an open set and $f \in D(G)$. If there is an $a \in G$ such that $\lim_{x\to a} f'(x)$ exists, then $\lim_{x\to a} f'(x) = f'(a)$.

Exercise 7.5. Suppose f is continuous on $[a,b]$ and f'' exists on (a,b). If there is an $x_0 \in (a,b)$ such that the line segment between $(a, f(a))$ and $(b, f(b))$ contains the point $(x_0, f(x_0))$, then there is a $c \in (a,b)$ such that $f''(c) = 0$.

Exercise 7.6. If $\Delta = \{f : f = F' \text{ for some } F : \mathbb{R} \to \mathbb{R}\}$, then Δ is closed under addition and scalar multiplication. (This shows the derivatives form a vector space.)

Exercise 7.7. If

$$f_1(x) = \begin{cases} 1/2, & x = 0 \\ \sin(1/x), & x \neq 0 \end{cases}$$

and

$$f_2(x) = \begin{cases} 1/2, & x = 0 \\ \sin(-1/x), & x \neq 0 \end{cases},$$

then at least one of f_1 and f_2 is not in Δ.

Exercise 7.8. Suppose f is differentiable everywhere and $f(x+y) = f(x)f(y)$ for all $x, y \in \mathbb{R}$. Show that $f'(x) = f'(0)f(x)$ and determine the value of $f'(0)$.

Exercise 7.9. If I is an open interval, f is differentiable on I and $a \in I$, then there is a sequence $a_n \in I \setminus \{a\}$ such that $a_n \to a$ and $f'(a_n) \to f'(a)$.

Exercise 7.10. Use the definition of the derivative to find $\dfrac{d}{dx}\sqrt{x}$.

Exercise 7.11. Let f be continuous on $[0, \infty)$ and differentiable on $(0, \infty)$. If $f(0) = 0$ and $|f'(x)| \leq |f(x)|$ for all $x > 0$, then $f(x) = 0$ for all $x \geq 0$.

Exercise 7.12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is such that f' is continuous on $[a, b]$. If there is a $c \in (a, b)$ such that $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.

Exercise 7.13. Prove or give a counter example: If f is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R} \setminus \{0\}$ with $\lim_{x \to 0} f'(x) = L$, then f is differentiable on $\mathbb{R}$.

Exercise 7.14. Let f be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = \alpha$ and $|f'(x)| < \beta$ for all $x \in (a, b)$, then calculate a bound for $f(b)$.

Exercise 7.15. Suppose that $f : (a, b) \to \mathbb{R}$ is differentiable and f' is bounded. If x_n is a sequence from (a, b) such that $x_n \to a$, then $f(x_n)$ converges.

Exercise 7.16. Let G be an open set and $f \in D(G)$. If there is an $a \in G$ such that $\lim_{x \to a} f'(x)$ exists, then $\lim_{x \to a} f'(x) = f'(a)$.

Exercise 7.17. Prove or give a counter example: If $f \in D((a, b))$ such that f' is bounded, then there is an $F \in C([a, b])$ such that $f = F$ on (a, b).

Exercise 7.18. Show that $f(x) = x^3 + 2x + 1$ is invertible on $\mathbb{R}$ and, if $g = f^{-1}$, then find $g'(1)$.

Exercise 7.19. Suppose that I is an open interval and that $f''(x) \geq 0$ for all $x \in I$. If $a \in I$, then show that the part of the graph of f on I is never below the tangent line to the graph at $(a, f(a))$.

Exercise 7.20. Suppose f is continuous on $[a, b]$ and f'' exists on (a, b). If there is an $x_0 \in (a, b)$ such that the line segment between $(a, f(a))$ and $(b, f(b))$ contains the point $(x_0, f(x_0))$, then there is a $c \in (a, b)$ such that $f''(c) = 0$.

Exercise 7.21. Let f be defined on a neighborhood of x.

(a) If $f''(x)$ exists, then

$$\lim_{h \to 0} \frac{f(x-h) - 2f(x) + f(x+h)}{h^2} = f''(x).$$

(b) Find a function f where this limit exists, but $f''(x)$ does not exist.

Exercise 7.22. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and is even, then f' is odd. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable everywhere and is odd, then f' is even.[7]

Exercise 7.23. Use $f(x) = \ln(1 + x)$ to prove the alternating harmonic series converges to $\ln 2$.

Exercise 7.24. Prove that

$$\left| \sin x - \left(x - \frac{x^3}{6} + \frac{x^5}{120} \right) \right| < \frac{1}{5040}$$

when $|x| \leq 1$.

Exercise 7.25. The exponential function e^x is not a polynomial.

[7] A function g is *even* if $g(-x) = g(x)$ for every x and it is *odd* if $g(-x) = -g(x)$ for every x. Even and odd functions are described as such because this is how $g(x) = x^n$ behaves when n is an even or odd integer, respectively.

Chapter 8

Integration

Contrary to the impression given by most calculus courses, there are many ways to define integration. The one given here is called the Riemann integral or the Riemann-Darboux integral, and it is the one most commonly presented to calculus students.

8.1 Partitions

A *partition* of the interval $[a, b]$ is a finite set $P \subset [a, b]$ such that $\{a, b\} \subset P$. The set of all partitions of $[a, b]$ is denoted $\text{part}([a, b])$. Basically, a partition should be thought of as a way to divide an interval into a finite number of subintervals by listing the points where it is divided.

If $P \in \text{part}([a, b])$, then the elements of P can be ordered in a list as $a = x_0 < x_1 < \cdots < x_n = b$. The adjacent points of this partition determine n compact intervals of the form $I_k^P = [x_{k-1}, x_k]$, $1 \le k \le n$. If the partition is understood from the context, we write I_k instead of I_k^P. It's clear these intervals only intersect at their common endpoints and there is no requirement they have the same length.

Since it's inconvenient to always list each part of a partition, we'll use the partition of the previous paragraph as the generic partition. Unless it's necessary within the context to specify some other form for a partition, assume any partition is the generic partition. (See Figure 8.1.)

If I is any interval, its length is written $|I|$. Using the notation of the previous paragraph, it follows that

$$\sum_{k=1}^{n} |I_k| = \sum_{k=1}^{n} (x_k - x_{k-1}) = x_n - x_0 = b - a.$$

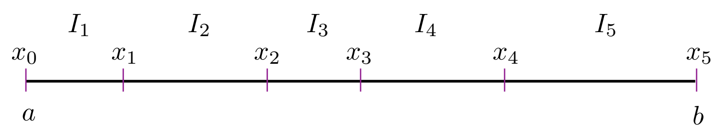

Figure 8.1: The generic partition with five subintervals.

The *norm* of a partition P is

$$\|P\| = \max\{|I_k^P| : 1 \le k \le n\}.$$

In other words, the norm of P is just the length of the longest subinterval determined by P. If $|I_k| = \|P\|$ for every I_k, then P is called a *regular* partition.

Suppose $P, Q \in \text{part}([a,b])$. If $P \subset Q$, then Q is called a *refinement* of P. When this happens, we write $P \ll Q$. In this case, it's easy to see that $P \ll Q$ implies $\|P\| \ge \|Q\|$. It also follows at once from the definitions that $P \cup Q \in \text{part}([a,b])$ with $P \ll P \cup Q$ and $Q \ll P \cup Q$. The partition $P \cup Q$ is called the *common refinement* of P and Q.

8.2 Riemann Sums

Let $f : [a,b] \to \mathbb{R}$ and $P \in \text{part}([a,b])$. Choose $x_k^* \in I_k$ for each k. The set $\{x_k^* : 1 \le k \le n\}$ is called a *selection* from P. The expression

$$\mathcal{R}(f, P, x_k^*) = \sum_{k=1}^{n} f(x_k^*)|I_k|$$

is the *Riemann sum* for f with respect to the partition P and selection x_k^*. The Riemann sum is the usual first step toward integration in a calculus course and can be visualized as the sum of the areas of rectangles with height $f(x_k^*)$ and width $|I_k|$ — as long as the rectangles are allowed to have negative area when $f(x_k^*) < 0$. (See Figure 8.2.)

Notice that given a particular function f and partition P, there are an uncountably infinite number of different possible Riemann sums, depending on the selection x_k^*. This sometimes makes working with Riemann sums quite complicated.

Example 8.1. Suppose $f : [a,b] \to \mathbb{R}$ is the constant function $f(x) = c$. If $P \in \text{part}([a,b])$ and $\{x_k^* : 1 \le k \le n\}$ is any selection from P, then

$$\mathcal{R}(f, P, x_k^*) = \sum_{k=1}^{n} f(x_k^*)|I_k| = c\sum_{k=1}^{n} |I_k| = c(b-a).$$

Example 8.2. Suppose $f(x) = x$ on $[a, b]$. Choose any $P \in \text{part}([a, b])$ where $\|P\| < 2(b-a)/n$. (Convince yourself this is always possible.[1]) Make two specific selections $l_k^* = x_{k-1}$ and $r_k^* = x_k$. If x_k^* is any other selection from P, then $l_k^* \le x_k^* \le r_k^*$ and the fact that f is increasing on $[a, b]$ gives

$$\mathcal{R}(f, P, l_k^*) \le \mathcal{R}(f, P, x_k^*) \le \mathcal{R}(f, P, r_k^*).$$

With this in mind, consider the following calculation.

$$\begin{aligned}
\mathcal{R}(f, P, r_k^*) - \mathcal{R}(f, P, l_k^*) &= \sum_{k=1}^{n} (r_k^* - l_k^*)|I_k| \qquad (8.1)\\
&= \sum_{k=1}^{n} (x_k - x_{k-1})|I_k| \\
&= \sum_{k=1}^{n} |I_k|^2 \\
&\le \sum_{k=1}^{n} \|P\|^2 \\
&= n\|P\|^2 \\
&< \frac{4(b-a)^2}{n}
\end{aligned}$$

[1] This is with the generic partition

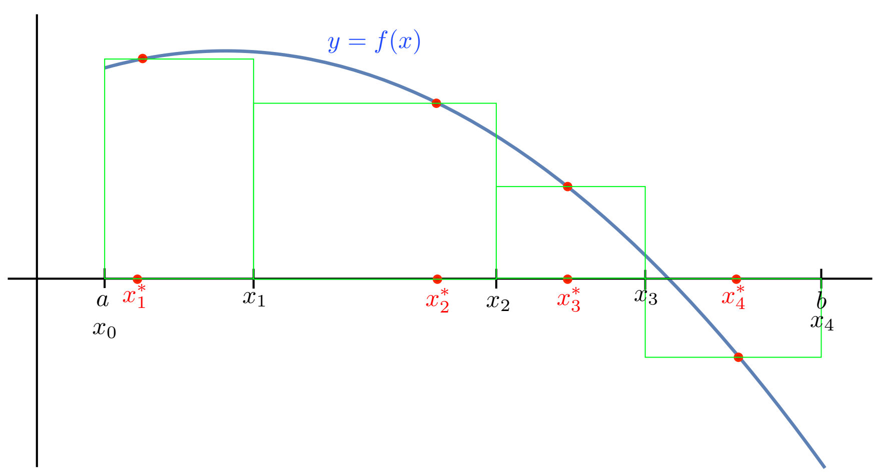

Figure 8.2: The Riemann sum $\mathcal{R}(f, P, x_k^*)$ is the sum of the areas of the rectangles in this figure. Notice the right-most rectangle has negative area because $f(x_4^*) < 0$.

This shows that if a partition is chosen with a small enough norm, all the Riemann sums for f over that partition will be close to each other.

In the special case when P is a regular partition, $|I_k| = (b-a)/n$, $r_k = a + k(b-a)/n$ and

$$\begin{aligned}
\mathcal{R}\left(f, P, r_k^*\right) &= \sum_{k=1}^{n} r_k |I_k| \\
&= \sum_{k=1}^{n} \left(a + \frac{k(b-a)}{n} \right) \frac{b-a}{n} \\
&= \frac{b-a}{n} \left(na + \frac{b-a}{n} \sum_{k=1}^{n} k \right) \\
&= \frac{b-a}{n} \left(na + \frac{b-a}{n} \frac{n(n+1)}{2} \right) \\
&= \frac{b-a}{2} \left(a\frac{n-1}{n} + b\frac{n+1}{n} \right).
\end{aligned}$$

In the limit as $n \to \infty$, this becomes the familiar formula $(b^2 - a^2)/2$, for the integral of $f(x) = x$ over $[a, b]$.

Definition 8.1. The function f is *Riemann integrable* on $[a, b]$, if there exists a number $\mathcal{R}(f)$ such that for all $\varepsilon > 0$ there is a $\delta > 0$ so that whenever $P \in \operatorname{part}([a, b])$ with $\|P\| < \delta$, then

$$\left|\mathcal{R}(f) - \mathcal{R}\left(f, P, x_k^*\right)\right| < \varepsilon$$

for any selection x_k^* from P.

Theorem 8.2. *If $f : [a, b] \to \mathbb{R}$ and $\mathcal{R}(f)$ exists, then $\mathcal{R}(f)$ is unique.*

Proof. Suppose $\mathcal{R}_1(f)$ and $\mathcal{R}_2(f)$ both satisfy the definition and $\varepsilon > 0$. For $i = 1, 2$ choose $\delta_i > 0$ so that whenever $\|P\| < \delta_i$, then

$$\left|\mathcal{R}_i(f) - \mathcal{R}\left(f, P, x_k^*\right)\right| < \varepsilon/2,$$

as in the definition above. If $P \in \operatorname{part}([a, b])$ so that $\|P\| < \delta_1 \wedge \delta_2$, then

$$|\mathcal{R}_1(f) - \mathcal{R}_2(f)| \le \left|\mathcal{R}_1(f) - \mathcal{R}\left(f, P, x_k^*\right)\right| + \left|\mathcal{R}_2(f) - \mathcal{R}\left(f, P, x_k^*\right)\right| < \varepsilon$$

and it follows $\mathcal{R}_1(f) = \mathcal{R}_2(f)$. □

Theorem 8.3. *If $f : [a, b] \to \mathbb{R}$ and $\mathcal{R}(f)$ exists, then f is bounded.*

Proof. Left as Exercise 2. □

8.3 Darboux Integration

As mentioned above, a difficulty with handling Riemann sums is there are so many different ways to choose partitions and selections that working with them is unwieldy. One way to resolve this problem was shown in Example 8.2, where it was easy to find largest and smallest Riemann sums associated with each partition. However, that's not always a straightforward calculation, so to use that idea, a little more care must be taken.

Definition 8.4. Let $f : [a, b] \to \mathbb{R}$ be bounded and $P \in \operatorname{part}([a, b])$. For each I_k determined by P, let

$$M_k = \operatorname{lub}\{f(x) : x \in I_k\} \quad \text{and} \quad m_k = \operatorname{glb}\{f(x) : x \in I_k\}.$$

The *upper and lower Darboux sums* for f on $[a, b]$ are

$$\overline{\mathcal{D}}(f, P) = \sum_{k=1}^{n} M_k |I_k| \quad \text{and} \quad \underline{\mathcal{D}}(f, P) = \sum_{k=1}^{n} m_k |I_k|.$$

The following theorem is the fundamental relationship between Darboux sums. Pay careful attention because it's the linchpin holding everything together!

Theorem 8.5. *If $f : [a, b] \to \mathbb{R}$ is bounded and $P, Q \in \operatorname{part}([a, b])$ with $P \ll Q$, then*

$$\underline{\mathcal{D}}(f, P) \le \underline{\mathcal{D}}(f, Q) \le \overline{\mathcal{D}}(f, Q) \le \overline{\mathcal{D}}(f, P).$$

Proof. Let P be the generic partition and let $Q = P \cup \{\overline{x}\}$, where $\overline{x} \in (x_{k_0-1}, x_{x_0})$ for some k_0. Clearly, $P \ll Q$. Let

$$\begin{aligned}
M_l &= \operatorname{lub}\{f(x) : x \in [x_{k_0-1}, \overline{x}]\} \\
m_l &= \operatorname{glb}\{f(x) : x \in [x_{k_0-1}, \overline{x}]\} \\
M_r &= \operatorname{lub}\{f(x) : x \in [\overline{x}, x_{k_0}]\} \\
m_r &= \operatorname{glb}\{f(x) : x \in [\overline{x}, x_{k_0}]\}
\end{aligned}$$

Then

$$m_{k_0} \le m_l \le M_l \le M_{k_0} \quad \text{and} \quad m_{k_0} \le m_r \le M_r \le M_{k_0}$$

so that

$$\begin{aligned}
m_{k_0}|I_{k_0}| &= m_{k_0}\left(|[x_{k_0-1},\overline{x}]| + |[\overline{x},x_{k_0}]|\right)\\
&\le m_l|[x_{k_0-1},\overline{x}]| + m_r|[\overline{x},x_{k_0}]|\\
&\le M_l|[x_{k_0-1},\overline{x}]| + M_r|[\overline{x},x_{k_0}]|\\
&\le M_{k_0}|[x_{k_0-1},\overline{x}]| + M_{k_0}|[\overline{x},x_{k_0}]|\\
&= M_{k_0}|I_{k_0}|.
\end{aligned}$$

This implies

$$\begin{aligned}
\underline{\mathcal{D}}(f,P) &= \sum_{k=1}^{n} m_k|I_k|\\
&= \sum_{k\neq k_0} m_k|I_k| + m_{k_0}|I_{k_0}|\\
&\le \sum_{k\neq k_0} m_k|I_k| + m_l|[x_{k_0-1},\overline{x}]| + m_r|[\overline{x},x_{k_0}]|\\
&= \underline{\mathcal{D}}(f,Q)\\
&\le \overline{\mathcal{D}}(f,Q)\\
&= \sum_{k\neq k_0} M_k|I_k| + M_l|[x_{k_0-1},\overline{x}]| + M_r|[\overline{x},x_{k_0}]|\\
&\le \sum_{k=1}^{n} M_k|I_k|\\
&= \overline{\mathcal{D}}(f,P)
\end{aligned}$$

The argument given above shows that the theorem holds if Q has one more point than P. Using induction, this same technique also shows the theorem holds when Q has an arbitrarily larger number of points than P. $\square$

The main lesson to be learned from Theorem 8.5 is that refining a partition causes the lower Darboux sum to increase and the upper Darboux sum to decrease. Moreover, if $P, Q \in \text{part}([a,b])$ and $f : [a,b] \to [-B,B]$, then,

$$-B(b-a) \le \underline{\mathcal{D}}(f,P) \le \underline{\mathcal{D}}(f,P\cup Q) \le \overline{\mathcal{D}}(f,P\cup Q) \le \overline{\mathcal{D}}(f,Q) \le B(b-a).$$

Therefore *every Darboux lower sum is less than or equal to every Darboux upper sum.* Consider the following definition with this in mind.

Definition 8.6. The *upper and lower Darboux integrals* of a bounded function $f : [a,b] \to \mathbb{R}$ are

$$\overline{\mathcal{D}}(f) = \text{glb}\left\{\overline{\mathcal{D}}(f,P) : P \in \text{part}([a,b])\right\}$$

and

$$\underline{\mathcal{D}}\,(f) = \operatorname{lub}\{\underline{\mathcal{D}}\,(f,P) : P \in \operatorname{part}([a,b])\},$$

respectively.

As a consequence of the observations preceding the definition, it follows that $\overline{\mathcal{D}}\,(f) \geq \underline{\mathcal{D}}\,(f)$ always. In the case $\overline{\mathcal{D}}\,(f) = \underline{\mathcal{D}}\,(f)$, the function is said to be *Darboux integrable* on $[a,b]$, and the common value is written $\mathcal{D}\,(f)$.

The following is obvious.

Corollary 8.7. *A bounded function $f : [a,b] \to \mathbb{R}$ is Darboux integrable if and only if for all $\varepsilon > 0$ there is a $P \in \operatorname{part}([a,b])$ such that $\overline{\mathcal{D}}\,(f,P) - \underline{\mathcal{D}}\,(f,P) < \varepsilon$.*

Which functions are Darboux integrable? The following corollary gives a first approximation to an answer.

Corollary 8.8. *If $f \in C([a,b])$, then $\mathcal{D}\,(f)$ exists.*

Proof. Let $\varepsilon > 0$. According to Corollary 6.31, f is uniformly continuous, so there is a $\delta > 0$ such that whenever $x, y \in [a,b]$ with $|x-y| < \delta$, then $|f(x) - f(y)| < \varepsilon/(b-a)$. Let $P \in \operatorname{part}([a,b])$ with $\|P\| < \delta$. By Corollary 6.23, in each subinterval I_i determined by P, there are $x_i^*, y_i^* \in I_i$ such that

$$f(x_i^*) = \operatorname{lub}\{f(x) : x \in I_i\} \quad \text{and} \quad f(y_i^*) = \operatorname{glb}\{f(x) : x \in I_i\}.$$

Since $|x_i^* - y_i^*| \leq |I_i| < \delta$, we see $0 \leq f(x_i^*) - f(y_i^*) < \varepsilon/(b-a)$, for $1 \leq i \leq n$. Then

$$\begin{aligned}
\overline{\mathcal{D}}\,(f) - \underline{\mathcal{D}}\,(f) &\leq \overline{\mathcal{D}}\,(f,P) - \underline{\mathcal{D}}\,(f,P) \\
&= \sum_{i=1}^{n} f(x_i^*)|I_i| - \sum_{i=1}^{n} f(y_i^*)|I_i| \\
&= \sum_{i=1}^{n} (f(x_i^*) - f(y_i^*))|I_i| \\
&< \frac{\varepsilon}{b-a} \sum_{i=1}^{n} |I_i| \\
&= \varepsilon
\end{aligned}$$

and the corollary follows. $\square$

This corollary should not be construed to imply that *only* continuous functions are Darboux integrable. In fact, the set of integrable functions is much more extensive than only the continuous functions. Consider the following example.

Example 8.3. Let f be the salt and pepper function of Example 6.15. It was shown that $C(f) = \mathbb{Q}^c$. We claim that f is Darboux integrable over any compact interval $[a, b]$.

To see this, let $\varepsilon > 0$ and $N \in \mathbb{N}$ so that $1/N < \varepsilon/2(b-a)$. Let

$$S = \{q_{k_i} : 1 \le i \le m\} = \{q_1, \ldots, q_N\} \cap [a, b]$$

and choose $P \in \operatorname{part}([a, b])$ such that $\|P\| < \varepsilon/2m$. Then

$$\begin{aligned}
\overline{\mathcal{D}}(f, P) &= \sum_{\ell=1}^{n} \operatorname{lub}\{f(x) : x \in I_\ell\}|I_\ell| \\
&= \sum_{S \cap I_\ell = \emptyset} \operatorname{lub}\{f(x) : x \in I_\ell\}|I_\ell| + \sum_{q_{k_i} \in I_\ell} \operatorname{lub}\{f(x) : x \in I_\ell\}|I_\ell| \\
&\le \frac{1}{N}(b-a) + m\|P\| \\
&< \frac{\varepsilon}{2(b-a)}(b-a) + m\frac{\varepsilon}{2m} \\
&= \varepsilon.
\end{aligned}$$

Since $f(x) = 0$ whenever $x \in \mathbb{Q}^c$, it follows that $\underline{\mathcal{D}}(f, P) = 0$. Therefore, $\overline{\mathcal{D}}(f) = \underline{\mathcal{D}}(f) = 0$ and $\mathcal{D}(f) = 0$.

8.4 The Integral

There are now two different definitions for the integral. It would be embarrassing, if they gave different answers. The following theorem shows they're really different sides of the same coin.[2]

Theorem 8.9. *Let $f : [a, b] \to \mathbb{R}$.*

(a) *$\mathcal{R}(f)$ exists iff $\mathcal{D}(f)$ exists.*

(b) *If $\mathcal{R}(f)$ exists, then $\mathcal{R}(f) = \mathcal{D}(f)$.*

Proof. (a) ($\Longrightarrow$) Suppose $\mathcal{R}(f)$ exists and $\varepsilon > 0$. By Theorem 8.3, f is bounded. Choose $P \in \operatorname{part}([a, b])$ such that

$$|\mathcal{R}(f) - \mathcal{R}(f, P, x_k^*)| < \varepsilon/4$$

[2]Theorem 8.9 shows that the two integrals presented here are the same. But, there are many other integrals, and not all of them are equivalent. For example, the well-known Lebesgue integral includes all Riemann integrable functions, but not all Lebesgue integrable functions are Riemann integrable. The Denjoy integral is another extension of the Riemann integral which is not the same as the Lebesgue integral. For more discussion of this, see [13].

for all selections x_k^* from P. From each I_k, choose $\overline{x}_k$ and $\underline{x}_k$ so that

$$M_k - f(\overline{x}_k) < \frac{\varepsilon}{4(b-a)} \quad \text{and} \quad f(\underline{x}_k) - m_k < \frac{\varepsilon}{4(b-a)}.$$

Then

$$\begin{aligned}\overline{\mathcal{D}}\,(f,P) - \mathcal{R}\,(f,P,\overline{x}_k) &= \sum_{k=1}^{n} M_k|I_k| - \sum_{k=1}^{n} f(\overline{x}_k)|I_k| \\ &= \sum_{k=1}^{n}(M_k - f(\overline{x}_k))|I_k| \\ &< \frac{\varepsilon}{4(b-a)}(b-a) = \frac{\varepsilon}{4}.\end{aligned}$$

In the same way,

$$\mathcal{R}\,(f,P,\underline{x}_k) - \underline{\mathcal{D}}\,(f,P) < \varepsilon/4.$$

Therefore,

$$\begin{aligned}&\overline{\mathcal{D}}\,(f) - \underline{\mathcal{D}}\,(f) \\ &\qquad = \operatorname{glb}\{\overline{\mathcal{D}}\,(f,Q) : Q \in \operatorname{part}([a,b])\} - \operatorname{lub}\{\underline{\mathcal{D}}\,(f,Q) : Q \in \operatorname{part}([a,b])\} \\ &\qquad \le \overline{\mathcal{D}}\,(f,P) - \underline{\mathcal{D}}\,(f,P) \\ &\qquad < \left(\mathcal{R}\,(f,P,\overline{x}_k) + \frac{\varepsilon}{4}\right) - \left(\mathcal{R}\,(f,P,\underline{x}_k) - \frac{\varepsilon}{4}\right) \\ &\qquad \le |\mathcal{R}\,(f,P,\overline{x}_k) - \mathcal{R}\,(f,P,\underline{x}_k)| + \frac{\varepsilon}{2} \\ &\qquad < |\mathcal{R}\,(f,P,\overline{x}_k) - \mathcal{R}\,(f)\,| + |\mathcal{R}\,(f) - \mathcal{R}\,(f,P,\underline{x}_k)\,| + \frac{\varepsilon}{2} \\ &\qquad < \varepsilon\end{aligned}$$

Since ε is an arbitrary positive number, this shows $\mathcal{D}\,(f)$ exists and equals $\mathcal{R}\,(f)$, which is part (b) of the theorem.

($\Longleftarrow$) Suppose $f : [a,b] \to [-B,B]$, $\mathcal{D}\,(f)$ exists and $\varepsilon > 0$. Since $\mathcal{D}\,(f)$ exists, there is a $P_1 \in \operatorname{part}([a,b])$, with points $a = p_0 < \cdots < p_m = b$, such that

$$\overline{\mathcal{D}}\,(f,P_1) - \underline{\mathcal{D}}\,(f,P_1) < \frac{\varepsilon}{2}.$$

Set $\delta = \varepsilon/8mB$. Choose $P \in \operatorname{part}([a,b])$ with $\|P\| < \delta$ and let $P_2 = P \cup P_1$. Since $P_1 \ll P_2$, according to Theorem 8.5,

$$\overline{\mathcal{D}}\,(f,P_2) - \underline{\mathcal{D}}\,(f,P_2) < \frac{\varepsilon}{2}.$$

Thinking of P as the generic partition, the interiors of its intervals (x_{i-1}, x_i) may or may not contain points of P_1. For $1 \leq i \leq n$, let

$$Q_i = \{x_{i-1}, x_i\} \cup (P_1 \cap (x_{i-1}, x_i)) \in \text{part}\,(I_i)\,.$$

If $P_1 \cap (x_{i-1}, x_i) = \emptyset$, then $\overline{\mathcal{D}}\,(f, P)$ and $\overline{\mathcal{D}}\,(f, P_2)$ have the term $M_i|I_i|$ in common because $Q_i = \{x_{i-1}, x_i\}$.

Otherwise, $P_1 \cap (x_{i-1}, x_i) \neq \emptyset$ and

$$\overline{\mathcal{D}}\,(f, Q_i) \geq -B\|P_2\| \geq -B\|P\| > -B\delta.$$

Since P_1 has $m-1$ points in (a, b), there are at most $m-1$ of the Q_i not contained in P.

This leads to the estimate

$$\overline{\mathcal{D}}\,(f, P) - \overline{\mathcal{D}}\,(f, P_2) = \overline{\mathcal{D}}\,(f, P) - \sum_{i=1}^{n} \overline{\mathcal{D}}\,(f, Q_i) < (m-1)2B\delta < \frac{\varepsilon}{4}.$$

In the same way,

$$\underline{\mathcal{D}}\,(f, P_2) - \underline{\mathcal{D}}\,(f, P) < (m-1)2B\delta < \frac{\varepsilon}{4}.$$

Putting these estimates together yields

$$\begin{aligned} &\overline{\mathcal{D}}\,(f, P) - \underline{\mathcal{D}}\,(f, P) = \\ &\left(\overline{\mathcal{D}}\,(f, P) - \overline{\mathcal{D}}\,(f, P_2)\right) + \left(\overline{\mathcal{D}}\,(f, P_2) - \underline{\mathcal{D}}\,(f, P_2)\right) + \left(\underline{\mathcal{D}}\,(f, P_2) - \underline{\mathcal{D}}\,(f, P)\right) \\ &\qquad\qquad\qquad\qquad < \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon \end{aligned}$$

This shows that, given $\varepsilon > 0$, there is a $\delta > 0$ so that $\|P\| < \delta$ implies

$$\overline{\mathcal{D}}\,(f, P) - \underline{\mathcal{D}}\,(f, P) < \varepsilon.$$

Since

$$\underline{\mathcal{D}}\,(f, P) \leq \mathcal{D}\,(f) \leq \overline{\mathcal{D}}\,(f, P) \text{ and } \underline{\mathcal{D}}\,(f, P) \leq \mathcal{R}\,(f, P, x_i^*) \leq \overline{\mathcal{D}}\,(f, P)$$

for every selection x_i^* from P, it follows that $|\mathcal{R}\,(f, P, x_i^*) - \mathcal{D}\,(f)\,| < \varepsilon$ when $\|P\| < \delta$. We conclude f is Riemann integrable and $\mathcal{R}\,(f) = \mathcal{D}\,(f)$. □

From Theorem 8.9, we are justified in using a single notation for both $\mathcal{R}\,(f)$ and $\mathcal{D}\,(f)$. The obvious choice is the familiar $\int_a^b f(x)\,dx$, or, more simply, $\int_a^b f$.

When proving statements about the integral, it's convenient to switch back and forth between the Riemann and Darboux formulations. Given $f : [a, b] \to \mathbb{R}$ the following three facts summarize much of what we know.

1. $\int_a^b f$ exists iff for all $\varepsilon > 0$ there is a $\delta > 0$ and an $\alpha \in \mathbb{R}$ such that whenever $P \in \text{part}([a,b])$ with $\|P\| < \delta$ and x_i^* is a selection from P, then $|\mathcal{R}(f, P, x_i^*) - \alpha| < \varepsilon$. In this case $\int_a^b f = \alpha$.

2. $\int_a^b f$ exists iff $\forall \varepsilon > 0 \exists P \in \text{part}([a,b]) \left(\overline{\mathcal{D}}(f,P) - \underline{\mathcal{D}}(f,P) < \varepsilon\right)$

3. For any $P \in \text{part}([a,b])$ and selection x_i^* from P,

$$\underline{\mathcal{D}}(f,P) \leq \mathcal{R}(f,P,x_i^*) \leq \overline{\mathcal{D}}(f,P).$$

8.5 The Cauchy Criterion

We now face a conundrum. In order to show that $\int_a^b f$ exists, we must know its value. It's often very hard to determine the value of an integral, even if the integral exists. We've faced this same situation before with sequences. The basic definition of convergence for a sequence, Definition 3.2, requires the limit of the sequence be known. The path out of the dilemma in the case of sequences was the Cauchy criterion for convergence, Theorem 3.20. The solution is the same here, with a Cauchy criterion for the existence of the integral.

Theorem 8.10 (Cauchy Criterion)**.** *Let $f : [a,b] \to \mathbb{R}$. The following statements are equivalent.*

(a) $\int_a^b f$ *exists.*

(b) *Given $\varepsilon > 0$ there exists $P \in \text{part}([a,b])$ such that if $P \ll Q_1$ and $P \ll Q_2$, then*

$$|\mathcal{R}(f, Q_1, x_k^*) - \mathcal{R}(f, Q_2, y_k^*)| < \varepsilon \tag{8.2}$$

for any selections from Q_1 and Q_2.

Proof. ($\Longrightarrow$) Assume $\int_a^b f$ exists. Let $\varepsilon > 0$. According to Definition 8.1, there is a $\delta > 0$ such that whenever $P \in \text{part}([a,b])$ with $\|P\| < \delta$, then $|\int_a^b f - \mathcal{R}(f,P,x_i^*)| < \varepsilon/2$ for every selection. If $P \ll Q_1$ and $P \ll Q_2$, then $\|Q_1\| < \delta$, $\|Q_2\| < \delta$ and a simple application of the triangle inequality shows

$$\begin{aligned}|\mathcal{R}(f, Q_1, x_k^*) &- \mathcal{R}(f, Q_2, y_k^*)| \\ &\leq \left|\mathcal{R}(f, Q_1, x_k^*) - \int_a^b f\right| + \left|\int_a^b f - \mathcal{R}(f, Q_2, y_k^*)\right| < \varepsilon.\end{aligned}$$

($\Longleftarrow$) Let $\varepsilon > 0$ and choose $P \in \operatorname{part}([a,b])$ satisfying (b) with $\varepsilon/2$ in place of ε.

We first claim that f is bounded. To see this, suppose it is not. Then it must be unbounded on an interval I_{k_0} determined by P. Fix a selection $\{x_k^* \in I_k : 1 \le k \le n\}$ and let $y_k^* = x_k^*$ for $k \ne k_0$ with $y_{k_0}^*$ any element of I_{k_0}. Then

$$\frac{\varepsilon}{2} > |\mathcal{R}(f,P,x_k^*) - \mathcal{R}(f,P,y_k^*)| = \left|f(x_{k_0}^*) - f(y_{k_0}^*)\right| |I_{k_0}|.$$

But, the right-hand side can be made bigger than $\varepsilon/2$ with an appropriate choice of $y_{k_0}^*$ because of the assumption that f is unbounded on I_{k_0}. This contradiction forces the conclusion that f is bounded.

Thinking of P as the generic partition and using m_k and M_k as usual with Darboux sums, for each k, choose $x_k^*, y_k^* \in I_k$ such that

$$M_k - f(x_k^*) < \frac{\varepsilon}{4n|I_k|} \text{ and } f(y_k^*) - m_k < \frac{\varepsilon}{4n|I_k|}.$$

With these selections,

$$\begin{aligned}
&\overline{\mathcal{D}}(f,P) - \underline{\mathcal{D}}(f,P)\\
&= \overline{\mathcal{D}}(f,P) - \mathcal{R}(f,P,x_k^*) + \mathcal{R}(f,P,x_k^*) - \mathcal{R}(f,P,y_k^*) + \mathcal{R}(f,P,y_k^*) - \underline{\mathcal{D}}(f,P)\\
&= \sum_{k=1}^n (M_k - f(x_k^*))|I_k| + \mathcal{R}(f,P,x_k^*) - \mathcal{R}(f,P,y_k^*) + \sum_{k=1}^n (f(y_k^*) - m_k)|I_k|\\
&\le \sum_{k=1}^n |M_k - f(x_k^*)|\,|I_k| + |\mathcal{R}(f,P,x_k^*) - \mathcal{R}(f,P,y_k^*)| + \left|\sum_{k=1}^n (f(y_k^*) - m_k)|I_k|\right|\\
&< \sum_{k=1}^n \frac{\varepsilon}{4n|I_k|}|I_k| + |\mathcal{R}(f,P,x_k^*) - \mathcal{R}(f,P,y_k^*)| + \sum_{k=1}^n \frac{\varepsilon}{4n|I_k|}|I_k|\\
&< \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} < \varepsilon
\end{aligned}$$

Corollary 8.7 implies $\mathcal{D}(f)$ exists and Theorem 8.9 finishes the proof. □

Corollary 8.11. *If $\int_a^b f$ exists and $[c,d] \subset [a,b]$, then $\int_c^d f$ exists.*

Proof. Let $P_0 = \{a,b,c,d\} \in \operatorname{part}([a,b])$ and $\varepsilon > 0$. Using Theorem 8.10, choose a partition P_ε such that $P_0 \ll P_\varepsilon$ and whenever $P_\varepsilon \ll P$ and $P_\varepsilon \ll P'$, then

$$|\mathcal{R}(f,P,x_k^*) - \mathcal{R}(f,P',y_k^*)| < \varepsilon.$$

Let $P_\varepsilon^1 = P_\varepsilon \cap [a, c]$, $P_\varepsilon^2 = P_\varepsilon \cap [c, d]$ and $P_\varepsilon^3 = P_\varepsilon \cap [d, b]$. Suppose $P_\varepsilon^2 \ll Q_1$ and $P_\varepsilon^2 \ll Q_2$. Then $P_\varepsilon^1 \cup Q_i \cup P_\varepsilon^3$ for $i = 1, 2$ are refinements of P_ε and

$$|\mathcal{R}(f, Q_1, x_k^*) - \mathcal{R}(f, Q_2, y_k^*)| = \\ |\mathcal{R}\left(f, P_\varepsilon^1 \cup Q_1 \cup P_\varepsilon^3, x_k^*\right) - \mathcal{R}\left(f, P_\varepsilon^1 \cup Q_2 \cup P_\varepsilon^3, y_k^*\right)| < \varepsilon$$

for any selections. An application of Theorem 8.10 shows $\int_a^b f$ exists. □

8.6 Properties of the Integral

Theorem 8.12. *If $\int_a^b f$ and $\int_a^b g$ both exist, then*

(a) *If $\alpha, \beta \in \mathbb{R}$, then $\int_a^b (\alpha f + \beta g)$ exists and*

$$\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g.$$

(b) *$\int_a^b fg$ exists.*

(c) *$\int_a^b |f|$ exists.*

Proof. (a) Let $\varepsilon > 0$. If $\alpha = 0$, in light of Example 8.1, it is clear αf is integrable. So, assume $\alpha \neq 0$, and choose a partition $P_f \in \text{part}([a, b])$ such that whenever $P_f \ll P$, then

$$\left| \mathcal{R}(f, P, x_k^*) - \int_a^b f \right| < \frac{\varepsilon}{2|\alpha|}.$$

Then

$$\begin{aligned} \left| \mathcal{R}(\alpha f, P, x_k^*) - \alpha \int_a^b f \right| &= \left| \sum_{k=1}^n \alpha f(x_k^*) |I_k| - \alpha \int_a^b f \right| \\ &= |\alpha| \left| \sum_{k=1}^n f(x_k^*) |I_k| - \int_a^b f \right| \\ &= |\alpha| \left| \mathcal{R}(f, P, x_k^*) - \int_a^b f \right| \\ &< |\alpha| \frac{\varepsilon}{2|\alpha|} \\ &= \frac{\varepsilon}{2}. \end{aligned}$$

This shows αf is integrable and $\int_a^b \alpha f = \alpha \int_a^b f$.

Assuming $\beta \neq 0$, in the same way, we can choose a $P_g \in \operatorname{part}([a,b])$ such that when $P_g \ll P$, then

$$\left| \mathcal{R}(g, P, x_k^*) - \int_a^b g \right| < \frac{\varepsilon}{2|\beta|}.$$

Let $P_\varepsilon = P_f \cup P_g$ be the common refinement of P_f and P_g, and suppose $P_\varepsilon \ll P$. Then

$$\begin{aligned} &|\mathcal{R}(\alpha f + \beta g, P, x_k^*) - \left(\alpha \int_a^b f + \beta \int_a^b g \right)| \\ &\qquad \leq |\alpha| \left| \mathcal{R}(f, P, x_k^*) - \int_a^b f \right| + |\beta| \left| \mathcal{R}(g, P, x_k^*) - \int_a^b g \right| < \varepsilon \end{aligned}$$

for any selection. This shows $\alpha f + \beta g$ is integrable and $\int_a^b (\alpha f + \beta g) = \alpha \int_a^b f + \beta \int_a^b g$.

(b) Claim: If $\int_a^b h$ exists, then so does $\int_a^b h^2$

To see this, suppose first that $0 \leq h(x) \leq M$ on $[a,b]$. If $M = 0$, the claim is trivially true, so suppose $M > 0$. Let $\varepsilon > 0$ and choose $P \in \operatorname{part}([a,b])$ such that

$$\overline{\mathcal{D}}(h, P) - \underline{\mathcal{D}}(h, P) \leq \frac{\varepsilon}{2M}.$$

For each $1 \leq k \leq n$, let

$$m_k = \operatorname{glb}\{h(x) : x \in I_k\} \leq \operatorname{lub}\{h(x) : x \in I_k\} = M_k.$$

Since $h \geq 0$,

$$m_k^2 = \operatorname{glb}\{h(x)^2 : x \in I_k\} \leq \operatorname{lub}\{h(x)^2 : x \in I_k\} = M_k^2.$$

Using this, we see

$$\begin{aligned} \overline{\mathcal{D}}(h^2, P) - \underline{\mathcal{D}}(h^2, P) &= \sum_{k=1}^n (M_k^2 - m_k^2)|I_k| \\ &= \sum_{k=1}^n (M_k + m_k)(M_k - m_k)|I_k| \\ &\leq 2M \left(\sum_{k=1}^n (M_k - m_k)|I_k| \right) \\ &= 2M \left(\overline{\mathcal{D}}(h, P) - \underline{\mathcal{D}}(h, P) \right) \\ &< \varepsilon. \end{aligned}$$

Therefore, h^2 is integrable when $h \geq 0$.

If h is not nonnegative, let $m = \text{glb}\,\{h(x) : a \leq x \leq b\}$. Then $h - m \geq 0$, and $h - m$ is integrable by (a). From the claim, $(h-m)^2$ is integrable. Since

$$h^2 = (h-m)^2 + 2mh - m^2,$$

it follows from (a) that h^2 is integrable.

Finally, $fg = \frac{1}{4}((f+g)^2 - (f-g)^2)$ is integrable by the claim and (a).

(c) Claim: If $h \geq 0$ is integrable, then so is $\sqrt{h}$.

To see this, let $\varepsilon > 0$ and choose $P \in \text{part}\,([a,b])$ such that

$$\overline{\mathcal{D}}\,(h,P) - \underline{\mathcal{D}}\,(h,P) < \varepsilon^2.$$

For each $1 \leq k \leq n$, let

$$m_k = \text{glb}\,\{\sqrt{h(x)} : x \in I_k\} \leq \text{lub}\,\{\sqrt{h(x)} : x \in I_k\} = M_k.$$

and define

$$A = \{k : M_k - m_k < \varepsilon\} \quad \text{and} \quad B = \{k : M_k - m_k \geq \varepsilon\}.$$

Then

$$\sum_{k \in A} (M_k - m_k)|I_k| < \varepsilon(b-a). \tag{8.3}$$

Using the fact that $m_k \geq 0$, we see that $M_k - m_k \leq M_k + m_k$, and

$$\begin{aligned}
\sum_{k \in B} (M_k - m_k)|I_k| &\leq \frac{1}{\varepsilon} \sum_{k \in B} (M_k + m_k)(M_k - m_k)|I_k| \qquad (8.4)\\
&= \frac{1}{\varepsilon} \sum_{k \in B} (M_k^2 - m_k^2)|I_k| \\
&\leq \frac{1}{\varepsilon} \left(\overline{\mathcal{D}}\,(h,P) - \underline{\mathcal{D}}\,(h,P)\right) \\
&< \varepsilon
\end{aligned}$$

Combining (8.3) and (8.4), it follows that

$$\overline{\mathcal{D}}\left(\sqrt{h}, P\right) - \underline{\mathcal{D}}\left(\sqrt{h}, P\right) < \varepsilon(b-a) + \varepsilon = \varepsilon((b-a)+1)$$

can be made arbitrarily small. Therefore, $\sqrt{h}$ is integrable.

Since $|f| = \sqrt{f^2}$ an application of (b) and the claim suffice to prove (c). $\square$

Theorem 8.13. *If $\int_a^b f$ exists, then*

(a) *If $f \geq 0$ on $[a,b]$, then $\int_a^b f \geq 0$.*

(b) $|\int_a^b f| \leq \int_a^b |f|$

(c) *If $a \leq c \leq b$, then $\int_a^b f = \int_a^c f + \int_c^b f$.*

Proof. (a) Since all the Riemann sums are nonnegative, this follows at once.

(b) It is always true that $|f| \pm f \geq 0$ and $|f| - f \geq 0$, so by (a), $\int_a^b (|f| + f) \geq 0$ and $\int_a^b (|f| - f) \geq 0$. Rearranging these shows $-\int_a^b f \leq \int_a^b |f|$ and $\int_a^b f \leq \int_a^b |f|$. Therefore, $|\int_a^b f| \leq \int_a^b |f|$, which is (b).

(c) By Corollary 8.11, all the integrals exist. Let $\varepsilon > 0$ and choose $P_l \in \text{part}([a,c])$ and $P_r \in \text{part}([c,b])$ such that whenever $P_l \ll Q_l$ and $P_r \ll Q_r$, then,

$$\left|\mathcal{R}(f, Q_l, x_k^*) - \int_a^c f\right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left|\mathcal{R}(f, Q_r, y_k^*) - \int_c^b f\right| < \frac{\varepsilon}{2}.$$

If $P = P_l \cup P_r$ and $Q = Q_l \cup Q_r$, then $P, Q \in \text{part}([a,b])$ and $P \ll Q$. The triangle inequality gives

$$\left|\mathcal{R}(f, Q, x_k^*) - \int_a^c f - \int_c^b f\right| < \varepsilon.$$

Since every refinement of P has the form $Q_l \cup Q_r$, part (c) follows. □

There's some notational trickery that can be played here. If $\int_a^b f$ exists, then we define $\int_b^a f = -\int_a^b f$. With this convention, it can be shown

$$\int_a^b f = \int_a^c f + \int_c^b f \tag{8.5}$$

no matter the order of a, b and c, as long as at least two of the integrals exist. (See Problem 6.)

8.7 The Fundamental Theorem of Calculus

Many students leave their first calculus course with the impression that integration and differentiation are inverse operations. While there is a lot of truth to this, the situation is a little more complicated. Presented below are two different versions of the Fundamental Theorem of Calculus with slightly different viewpoints of the relation between the two operations.

Theorem 8.14 (Fundamental Theorem of Calculus 1)**.** *Suppose $f, F : [a, b] \to \mathbb{R}$ satisfy*

(a) $\int_a^b f$ *exists*

(b) $F \in C([a, b]) \cap D((a, b))$

(c) $F'(x) = f(x)$, $\forall x \in (a, b)$

Then $\int_a^b f = F(b) - F(a)$.

Proof. Let $\varepsilon > 0$. According to (a) and Definition 8.1, $P \in \text{part}([a, b])$ can be chosen such that

$$\left| \mathcal{R}(f, P, x_k^*) - \int_a^b f \right| < \varepsilon.$$

for every selection from P. On each interval $[x_{k-1}, x_k]$ determined by P, the function F satisfies the conditions of the Mean Value Theorem. (See Corollary 7.13.) Therefore, for each k, there is an $c_k \in (x_{k-1}, x_k)$ such that

$$F(x_k) - F(x_{k-1}) = F'(c_k)(x_k - x_{k-1}) = f(c_k)|I_k|.$$

So,

$$\begin{aligned}
\left| \int_a^b f - (F(b) - F(a)) \right| &= \left| \int_a^b f - \sum_{k=1}^n (F(x_k) - F(x_{k-1}) \right| \\
&= \left| \int_a^b f - \sum_{k=1}^n f(c_k)|I_k| \right| \\
&= \left| \int_a^b f - \mathcal{R}(f, P, c_k) \right| \\
&< \varepsilon
\end{aligned}$$

and the theorem follows. □

Example 8.4. The Fundamental Theorem of Calculus can be used to give a different form of Taylor's theorem. As in Theorem 7.18, suppose f and its first $n + 1$ derivatives exist on $[a, b]$ and $\int_a^b f^{(n+1)}$ exists. There is a function $R_f(n, x, t)$ such that

$$R_f(n, x, t) = f(x) - \sum_{k=0}^n \frac{f^{(k)}(t)}{k!}(x - t)^k$$

for $a \le t \le b$. Differentiating both sides of the equation with respect to t, note that the right-hand side telescopes, so the result is

$$\frac{d}{dt}R_f(n,x,t) = -\frac{(x-t)^n}{n!}f^{(n+1)}(t).$$

Using Theorem 8.14 and the fact that $R_f(n,x,x) = 0$ gives

$$\begin{aligned} R_f(n,x,c) &= R_f(n,x,c) - R_f(n,x,x) \\ &= \int_x^c \frac{d}{dt}R_f(n,x,t)\,dt \\ &= \int_c^x \frac{(x-t)^n}{n!}f^{(n+1)}(t)\,dt, \end{aligned}$$

which is the integral form of the remainder from Taylor's formula.

Corollary 8.15 (Integration by Parts)**.** *If $f, g \in C([a,b]) \cap D((a,b))$ and both $f'g$ and fg' are integrable on $[a,b]$, then*

$$\int_a^b fg' + \int_a^b f'g = f(b)g(b) - f(a)g(a).$$

Proof. Use Theorems 7.3(c) and 8.14. □

Suppose $\int_a^b f$ exists. By Corollary 8.11, f is integrable on every interval $[a,x]$, for $x \in [a,b]$. This allows us to define a function $F : [a,b] \to \mathbb{R}$ as $F(x) = \int_a^x f$, called the *indefinite integral* of f on $[a,b]$.

Theorem 8.16 (Fundamental Theorem of Calculus 2)**.** *Let f be integrable on $[a,b]$ and F be the indefinite integral of f. Then $F \in C([a,b])$ and $F'(x) = f(x)$ whenever $x \in C(f) \cap (a,b)$.*

Proof. To show $F \in C([a,b])$, let $x_0 \in [a,b]$ and $\varepsilon > 0$. Since $\int_a^b f$ exists, there is an $M > \operatorname{lub}\{|f(x)| : a \le x \le b\}$. Choose $0 < \delta < \varepsilon/M$ and $x \in (x_0 - \delta, x_0 + \delta) \cap [a,b]$. Then

$$|F(x) - F(x_0)| = \left| \int_{x_0}^x f \right| \le M|x - x_0| < M\delta < \varepsilon$$

and $x_0 \in C(F)$.

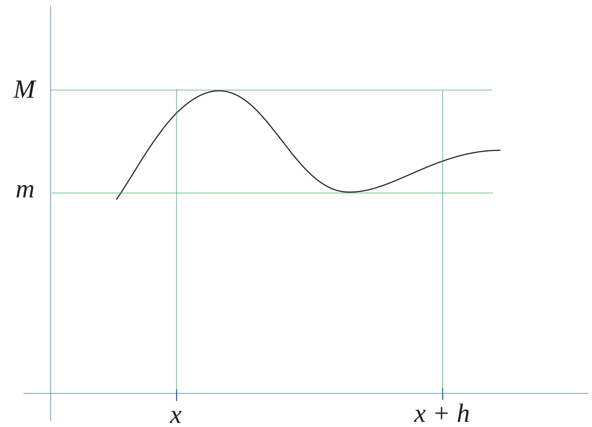

Figure 8.3: This figure illustrates a "box" argument showing $\lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f = f(x)$.

Let $x_0 \in C(f)\cap(a,b)$ and $\varepsilon > 0$. There is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0+\delta) \subset (a,b)$ implies $|f(x) - f(x_0)| < \varepsilon$. If $0 < h < \delta$, then

$$\begin{aligned}\left|\frac{F(x_0+h)-F(x_0)}{h} - f(x_0)\right| &= \left|\frac{1}{h}\int_{x_0}^{x_0+h} f - f(x_0)\right| \\ &= \left|\frac{1}{h}\int_{x_0}^{x_0+h} (f(t) - f(x_0))\,dt\right| \\ &\le \frac{1}{h}\int_{x_0}^{x_0+h} |f(t)-f(x_0)|\,dt \\ &< \frac{1}{h}\int_{x_0}^{x_0+h} \varepsilon\,dt \\ &= \varepsilon.\end{aligned}$$

This shows $F'_+(x_0) = f(x_0)$. It can be shown in the same way that $F'_-(x_0) = f(x_0)$. Therefore $F'(x_0) = f(x_0)$. $\square$

The right picture makes Theorem 8.16 almost obvious. Consider Figure 8.3. Suppose $x \in C(f)$ and $\varepsilon > 0$. There is a $\delta > 0$ such that

$$f((x-d, x+d)\cap[a,b]) \subset (f(x) - \varepsilon/2, f(x)+\varepsilon/2).$$

Let

$$m = \text{glb}\,\{fy : |x-y| < \delta\} \le \text{lub}\,\{fy : |x-y| < \delta\} = M.$$

Apparently $M - m < \varepsilon$ and for $0 < h < \delta$,

$$mh \le \int_x^{x+h} f \le Mh \implies m \le \frac{F(x+h) - F(x)}{h} \le M.$$

Since $M - m \to 0$ as $h \to 0$, a "squeezing" argument shows

$$\lim_{h \downarrow 0} \frac{F(x+h) - F(x)}{h} = f(x).$$

A similar argument establishes the limit from the left and $F'(x) = f(x)$.

Example 8.5. The usual definition of the natural logarithm function depends on the Fundamental Theorem of Calculus. Recall for $x > 0$,

$$\ln(x) = \int_1^x \frac{1}{t}\,dt.$$

Since $f(t) = 1/t$ is continuous on $(0, \infty)$, Theorem 8.16 shows

$$\frac{d}{dx}\ln(x) = \frac{1}{x}.$$

It should also be noted that the notational convention mentioned above equation (8.5) is used to get $\ln(x) < 0$ when $0 < x < 1$.

It's easy to read too much into the Fundamental Theorem of Calculus. We are tempted to start thinking of integration and differentiation as opposites of each other. But, this is far from the truth. The operations of integration and antidifferentiation are different operations, that happen to sometimes be tied together by the Fundamental Theorem of Calculus. Consider the following examples.

Example 8.6. Let

$$f(x) = \begin{cases} |x|/x, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

It's easy to prove that f is integrable over any compact interval, and that $F(x) = \int_{-1}^x f = |x| - 1$ is an indefinite integral of f. But, F is not differentiable at $x = 0$ and f is not a derivative, according to Theorem 7.17.

Example 8.7. Let

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x^2}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

It's straightforward to show that f is differentiable and

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x^2} - \frac{2}{x} \cos \frac{1}{x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Since f' is unbounded near $x = 0$, it follows from Theorem 8.3 that f' is not integrable over any interval containing 0.

Example 8.8. Let f be the salt and pepper function of Example 6.15. It was shown in Example 8.3 that $\int_a^b f = 0$ on any interval $[a, b]$. If $F(x) = \int_0^x f$, then $F(x) = 0$ for all x and $F' = f$ only on $C(f) = \mathbb{Q}^c$. In particular, F' and f disagree on a dense subset of $[a, b]$.

8.8 Change of Variables

Integration by substitution works side-by-side with the Fundamental Theorem of Calculus in the integration section of any calculus course. Most of the time calculus books require all functions in sight to be continuous. In that case, a substitution theorem is an easy consequence of the Fundamental Theorem and the Chain Rule. (See Exercise 16.) More general statements are true, but they are harder to prove.

Theorem 8.17. *If f and g are functions such that*

(a) *g is strictly monotone on $[a, b]$,*

(b) *g is continuous on $[a, b]$,*

(c) *g is differentiable on (a, b), and*

(d) *both $\int_{g(a)}^{g(b)} f$ and $\int_a^b (f \circ g) g'$ exist,*

then

$$\int_{g(a)}^{g(b)} f = \int_a^b (f \circ g) g'. \tag{8.6}$$

Proof. Suppositions (a) and (b) show g is a bijection from $[a, b]$ to an interval $[c, d]$. The correspondence between the endpoints depends on whether g is increasing or decreasing.

Let $\varepsilon > 0$.

From (d) and Definition 8.1, there is a $\delta_1 > 0$ such that whenever $P \in \text{part}([a,b])$ with $\|P\| < \delta_1$, then

$$\left| \mathcal{R}\left((f\circ g)g', P, x_i^*\right) - \int_a^b (f\circ g)g' \right| < \frac{\varepsilon}{2} \tag{8.7}$$

for any selection from P. Choose $P_1 \in \text{part}([a,b])$ such that $\|P_1\| < \delta_1$.

Using the same argument, there is a $\delta_2 > 0$ such that whenever $Q \in \text{part}([c,d])$ with $\|Q\| < \delta_2$, then

$$\left| \mathcal{R}\left(f, Q, x_i^*\right) - \int_c^d f \right| < \frac{\varepsilon}{2} \tag{8.8}$$

for any selection from Q. As above, choose $Q_1 \in \text{part}([c,d])$ such that $\|Q_1\| < \delta_2$.

Setting $P_2 = P_1 \cup \{g^{-1}(x) : x \in Q_1\}$ and $Q_2 = Q_1 \cup \{g(x) : x \in P_1\}$, it is apparent that $P_1 \ll P_2$, $Q_1 \ll Q_2$, $\|P_2\| \le \|P_1\| < \delta_1$, $\|Q_2\| \le \|Q_1\| < \delta_2$ and g is a bijection between P_2 and Q_2. From (8.7) and (8.8), it follows that

$$\begin{gathered} \left| \int_a^b (f\circ g)g' - \mathcal{R}\left((f\circ g)g', P_2, x_i^*\right) \right| < \frac{\varepsilon}{2} \\ \text{and} \\ \left| \int_c^d f - \mathcal{R}\left(f, Q_2, y_i^*\right) \right| < \frac{\varepsilon}{2} \end{gathered} \tag{8.9}$$

for any selections from P_2 and Q_2.

First, assume g is strictly increasing. Since g is a bijection between $[a,b]$ and $[c,d]$, it's clear P_2 and Q_2 have the same number of points. Label the points of P_2 as $a = x_1 < x_2 < \cdots < x_n = b$ and those of Q_2 as $c = y_0 < y_1 < \cdots < y_n = d$.

From (b), (c) and the Mean Value Theorem, for each i, choose $c_i \in (x_{i-1}, x_i)$ such that

$$g(x_i) - g(x_{i-1}) = g'(c_i)(x_i - x_{i-1}). \tag{8.10}$$

Notice that $\{c_i : 1 \le i \le n\}$ is a selection from P_2. and $g(x_i) = y_i$ for $0 \le i \le n$ and $g(c_i) \in (y_{i-1}, y_i)$ for $0 < i \le n$, so $g(c_i)$ is a selection from Q_2.

$$\begin{aligned} &\left| \int_{g(a)}^{g(b)} f - \int_a^b (f\circ g)g' \right| \\ &\qquad = \left| \int_{g(a)}^{g(b)} f - \mathcal{R}\left(f, Q_2, g(c_i)\right) + \mathcal{R}\left(f, Q_2, g(c_i)\right) - \int_a^b (f\circ g)g' \right| \\ &\qquad\quad \le \left| \int_{g(a)}^{g(b)} f - \mathcal{R}\left(f, Q_2, g(c_i)\right) \right| + \left| \mathcal{R}\left(f, Q_2, g(c_i)\right) - \int_a^b (f\circ g)g' \right| \end{aligned}$$

Use the triangle inequality, (8.9) and expand the second Riemann sum.

$$< \frac{\varepsilon}{2} + \left| \sum_{i=1}^{n} f(g(c_i))\,(g(x_i) - g(x_{i-1})) - \int_a^b (f \circ g)g' \right|$$

Apply the Mean Value Theorem, as in (8.10), and then use (8.9).

$$\begin{aligned}
&= \frac{\varepsilon}{2} + \left| \sum_{i=1}^{n} f(g(c_i))g'(c_i)\,(x_i - x_{i-1}) - \int_a^b (f \circ g)g' \right| \\
&= \frac{\varepsilon}{2} + \left| \mathcal{R}\left((f \circ g)g', P_2, c_i\right) - \int_a^b (f \circ g)g' \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

and (8.6) follows.

Now assume g is strictly decreasing on $[a, b]$. Labeling $y_i = g(x_i)$, as above, the labeling is a little trickier because $d = y_0 > \cdots > y_n = c$. With this in mind, the proof is much the same as above, except there's extra bookkeeping required to keep track of the signs. From the Mean Value Theorem,

$$\begin{aligned}
y_i - y_{i-1} &= g(x_i) - g(x_{i-1}) \\
&= -(g(x_i) - g(x_{i-1})) \\
&= -g'(c_i)(x_i - x_{i-1}),
\end{aligned} \tag{8.11}$$

where $c_i \in (x_{i-1}, x_i)$ is as above. The rest of the proof is much like the case when g is increasing.

$$\begin{aligned}
&\left| \int_{g(a)}^{g(b)} f - \int_a^b (f \circ g)g' \right| \\
&\quad = \left| -\int_{g(b)}^{g(a)} f + \mathcal{R}(f, Q_2, g(c_i)) - \mathcal{R}(f, Q_2, g(c_i)) - \int_a^b (f \circ g)g' \right| \\
&\quad \le \left| -\int_{g(a)}^{g(b)} f + \mathcal{R}(f, Q_2, g(c_i)) \right| + \left| -\mathcal{R}(f, Q_2, g(c_i)) - \int_a^b (f \circ g)g' \right|
\end{aligned}$$

Use (8.9), expand the second Riemann sum and apply (8.11).

$$\begin{aligned}
&< \frac{\varepsilon}{2} + \left| -\sum_{i=1}^{n} f(g(c_i))(y_i - y_{i-1}) - \int_a^b (f \circ g)g' \right| \\
&= \frac{\varepsilon}{2} + \left| \sum_{i=1}^{n} f(g(c_i))g'(c_i)(x_{i+1} - x_i) - \int_a^b (f \circ g)|g'| \right|
\end{aligned}$$

Use (8.9).

$$
\begin{aligned}
&= \frac{\varepsilon}{2} + \left| \mathcal{R}\left((f \circ g)g', P_2, c_i\right) - \int_a^b (f \circ g)g' \right| \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

The theorem has been proved. $\square$

Example 8.9. Suppose we want to calculate $\int_{-1}^{1} \sqrt{1-x^2}\,dx$. Using the notation of Theorem 8.17, let $f(x) = \sqrt{1-x^2}$, $g(x) = \sin x$ and $[a,b] = [-\pi/2, \pi/2]$. In this case, g is an increasing function. Then (8.6) becomes

$$
\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}\,dx &= \int_{\sin(-\pi/2)}^{\sin(\pi/2)} \sqrt{1-x^2}\,dx \\
&= \int_{-\pi/2}^{\pi/2} \sqrt{1-\sin^2 x}\cos x\,dx \\
&= \int_{-\pi/2}^{\pi/2} \cos^2 x\,dx \\
&= \frac{\pi}{2}.
\end{aligned}
$$

On the other hand, it can also be done with a decreasing function. If $g(x) = \cos x$ and $[a,b] = [0,\pi]$, then

$$
\begin{aligned}
\int_{-1}^{1} \sqrt{1-x^2}\,dx &= \int_{\cos\pi}^{\cos 0} \sqrt{1-x^2}\,dx \\
&= -\int_{\cos 0}^{\cos\pi} \sqrt{1-x^2}\,dx \\
&= -\int_0^{\pi} \sqrt{1-\cos^2 x}(-\sin x)\,dx \\
&= \int_0^{\pi} \sqrt{1-\cos^2 x}\,\sin x\,dx \\
&= \int_0^{\pi} \sin^2 x\,dx \\
&= \frac{\pi}{2}
\end{aligned}
$$

8.9 Integral Mean Value Theorems

Theorem 8.18. *Suppose $f, g : [a,b] \to \mathbb{R}$ are such that*

(a) $g(x) \ge 0$ *on* $[a, b]$,

(b) f *is bounded and* $m \le f(x) \le M$ *for all* $x \in [a, b]$, *and*

(c) $\int_a^b f$ *and* $\int_a^b fg$ *both exist.*

There is a $c \in [m, M]$ *such that*

$$\int_a^b fg = c \int_a^b g.$$

Proof. Obviously,

$$m \int_a^b g \le \int_a^b fg \le M \int_a^b g. \tag{8.12}$$

If $\int_a^b g = 0$, we're done. Otherwise, let

$$c = \frac{\int_a^b fg}{\int_a^b g}.$$

Then $\int_a^b fg = c \int_a^b g$ and from (8.12), it follows that $m \le c \le M$. ☐

Corollary 8.19. *Let* f *and* g *be as in Theorem 8.18, but additionally assume* f *is continuous. Then there is a* $c \in (a, b)$ *such that*

$$\int_a^b fg = f(c) \int_a^b g.$$

Proof. This follows from Theorem 8.18 and Corollaries 6.23 and 6.26. ☐

Theorem 8.20. *Suppose* $f, g : [a, b] \to \mathbb{R}$ *are such that*

(a) $g(x) \ge 0$ *on* $[a, b]$,

(b) f *is bounded and* $m \le f(x) \le M$ *for all* $x \in [a, b]$, *and*

(c) $\int_a^b f$ *and* $\int_a^b fg$ *both exist.*

There is a $c \in [a, b]$ *such that*

$$\int_a^b fg = m \int_a^c g + M \int_c^b g.$$

Proof. For $a \le x \le b$ let

$$G(x) = m \int_a^x g + M \int_x^b g.$$

By Theorem 8.16, $G \in C([a,b])$ and

$$\operatorname{glb} G \le G(b) = m \int_a^b g \le \int_a^b fg \le M \int_a^b g = G(a) \le \operatorname{lub} G.$$

Now, apply Corollary 6.26 to find c where $G(c) = \int_a^b fg$. $\square$

8.10 Exercises

Exercise 8.1. Let $f : [a,b] \to \mathbb{R}$ be an unbounded function and $P \in \operatorname{part}([a,b])$. Prove

$$\{\mathcal{R}(f, P, x_i^*) : x_i^* \text{ is a selection from } P\}$$

is an unbounded set.

Exercise 8.2. If $f : [a,b] \to \mathbb{R}$ and $\mathcal{R}(f)$ exists, then f is bounded.

Exercise 8.3. Let

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}.$$

(a) Use Definition 8.1 to show f is not integrable on any interval.

(b) Use Definition 8.6 to show f is not integrable on any interval.

Exercise 8.4. Suppose $\int_a^b f$ exists, and $\varepsilon > 0$. Prove there is a $\delta > 0$ such that when $[c,d] \subset [a,b]$ with $d - c < \delta$, then $\left| \int_c^d f \right| < \varepsilon$.

Exercise 8.5. Calculate $\int_2^5 x^2$ using the definition of integration.

Exercise 8.6. If at least two of the integrals exist, then

$$\int_a^b f = \int_a^c f + \int_c^b f$$

no matter the order of a, b and c.

Exercise 8.7. If $\alpha > 0$, $f : [a,b] \to [\alpha,\beta]$ and $\int_a^b f$ exists, then $\int_a^b 1/f$ exists.

Exercise 8.8. If $f : [a,b] \to [0,\infty)$ is continuous and $\mathcal{D}(f) = 0$, then $f(x) = 0$ for all $x \in [a,b]$.

Exercise 8.9. If $\int_a^b f$ exists, then $\lim_{x \downarrow a} \int_x^b f = \int_a^b f$.

Exercise 8.10. If f is monotone on $[a,b]$, then $\int_a^b f$ exists.

Exercise 8.11. (a) Prove that for $n \in \mathbb{N}$,

$$\sum_{k=1}^{n} \frac{1}{k+1} < \ln(n+1) < \sum_{k=1}^{n} \frac{1}{k}.$$

(b) Prove that the sequence

$$\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

converges.[3]

Exercise 8.12. If f and g are integrable on $[a,b]$, then

$$\left| \int_a^b fg \right| \leq \left[\left(\int_a^b f^2 \right) \left(\int_a^b g^2 \right) \right]^{1/2}.$$

(Hint: Expand $\int_a^b (xf+g)^2$ as a quadratic with variable x.)[4]

Exercise 8.13. If $f : [a,b] \to [0,\infty)$ is continuous, then there is a $c \in [a,b]$ such that

$$f(c) = \left(\frac{1}{b-a} \int_a^b f^2 \right)^{1/2}.$$

Exercise 8.14. If $f(x) = \displaystyle\int_1^x \frac{dt}{t}$ for $x > 0$, then $f(xy) = f(x) + f(y)$ for $x, y > 0$.

[3] Its limit is the Euler-Mascheroni constant $\gamma \approx 0.57722$.

[4] This is variously called the Cauchy inequality, Cauchy-Schwarz inequality, or the Cauchy-Schwarz-Bunyakowsky inequality. Rearranging the last one, some people now call it the CBS inequality.

Exercise 8.15. If $f(x) = \ln(|x|)$ for $x \neq 0$, then $f'(x) = 1/x$.

Exercise 8.16. In the statement of Theorem 8.17, make the additional assumptions that f and g' are both continuous. Use the Fundamental Theorem of Calculus to give an easier proof.

Exercise 8.17. Find a function $f : [a, b] \to \mathbb{R}$ such that

(a) f is continuous on $[c, b]$ for all $c \in (a, b]$,

(b) $\lim_{x \downarrow a} \int_x^b f = 0$, and

(c) $\lim_{x \downarrow a} f(x)$ does not exist.

Exercise 8.18. Find a bounded function solving Exercise 17.

Chapter 9

Sequences of Functions

9.1 Pointwise Convergence

We have accumulated much experience working with sequences of numbers. The next level of complexity is sequences of functions. This chapter explores several ways that sequences of functions can converge to another function. The basic starting point is contained in the following definitions.

Definition 9.1. Suppose $S \subset \mathbb{R}$ and for each $n \in \mathbb{N}$ there is a function $f_n : S \to \mathbb{R}$. The collection $\{f_n : n \in \mathbb{N}\}$ is a *sequence* of functions defined on S.

For each fixed $x \in S$, $f_n(x)$ is a sequence of numbers, and it makes sense to ask whether this sequence converges. If $f_n(x)$ converges for each $x \in S$, a new function $f : S \to \mathbb{R}$ is defined by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

The function f is called the *pointwise limit* of the sequence f_n, or, equivalently, it is said f_n *converges pointwise* to f. This is abbreviated $f_n \xrightarrow{S} f$, or simply $f_n \to f$, if the domain is clear from the context.

Example 9.1. Let

$$f_n(x) = \begin{cases} 0, & x < 0 \\ x^n, & 0 \le x < 1 \\ 1, & x \ge 1 \end{cases}.$$

Then $f_n \to f$ where

$$f(x) = \begin{cases} 0, & x < 1 \\ 1, & x \ge 1 \end{cases}.$$

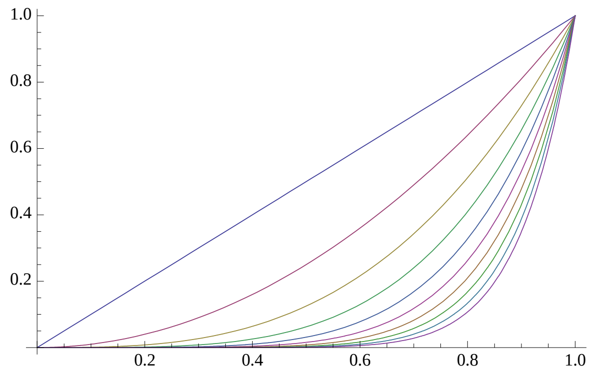

Figure 9.1: The first ten functions from the sequence of Example 9.1.

(See Figure 9.1.) This example shows that a pointwise limit of continuous functions need not be continuous.

Example 9.2. For each $n \in \mathbb{N}$, define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \frac{nx}{1 + n^2x^2}.$$

(See Figure 9.2.) Clearly, each f_n is an odd function and $\lim_{|x|\to\infty} f_n(x) = 0$. A bit of calculus shows that $f_n(1/n) = 1/2$ and $f_n(-1/n) = -1/2$ are the extreme values of f_n. Finally, if $x \neq 0$,

$$|f_n(x)| = \left|\frac{nx}{1 + n^2x^2}\right| < \left|\frac{nx}{n^2x^2}\right| = \left|\frac{1}{nx}\right|$$

implies $f_n \to 0$. This example shows that functions can remain bounded away from 0 and still converge pointwise to 0.

Example 9.3. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} 2^{2n+4}x - 2^{n+3}, & \frac{1}{2^{n+1}} < x < \frac{3}{2^{n+2}} \\ -2^{2n+4}x + 2^{n+4}, & \frac{3}{2^{n+2}} \leq x < \frac{1}{2^n} \\ 0, & \text{otherwise} \end{cases}$$

To figure out what this looks like, it might help to look at Figure 9.3.

The graph of f_n is a piecewise linear function supported on $[1/2^{n+1}, 1/2^n]$ and the area under the isosceles triangle of the graph over this interval is 1. Therefore, $\int_0^1 f_n = 1$ for all n.

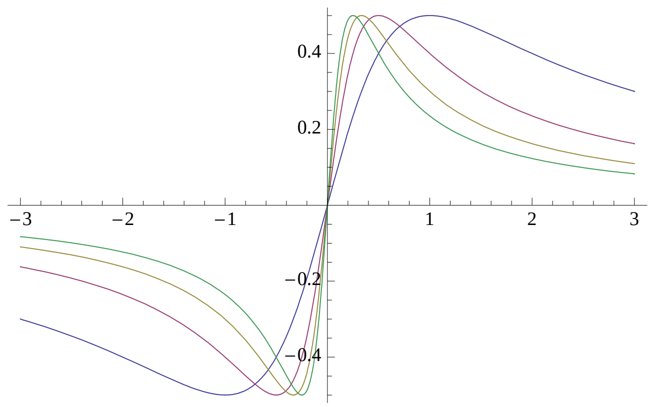

Figure 9.2: The first four functions from the sequence of Example 9.2.

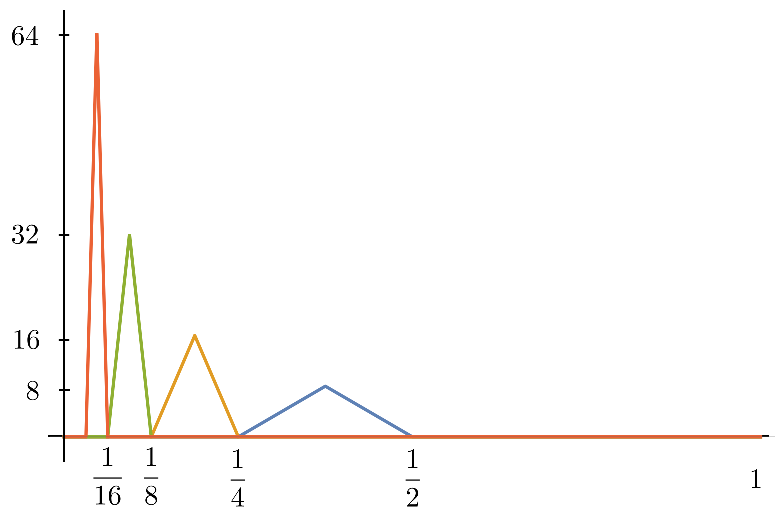

Figure 9.3: The first four functions $f_n \to 0$ from the sequence of Example 9.3.

If $x > 0$, then whenever $x > 1/2^n$, we have $f_n(x) = 0$. From this it follows that $f_n \to 0$.

The lesson to be learned from this example is that it may not be true that $\lim_{n\to\infty} \int_0^1 f_n = \int_0^1 \lim_{n\to\infty} f_n$.

Example 9.4. Define $f_n : \mathbb{R} \to \mathbb{R}$ by

$$f_n(x) = \begin{cases} \frac{n}{2}x^2 + \frac{1}{2n}, & |x| \le \frac{1}{n} \\ |x|, & |x| > \frac{1}{n} \end{cases}.$$

(See Figure 9.4.) The parabolic section in the center was chosen so $f_n(\pm 1/n) =$

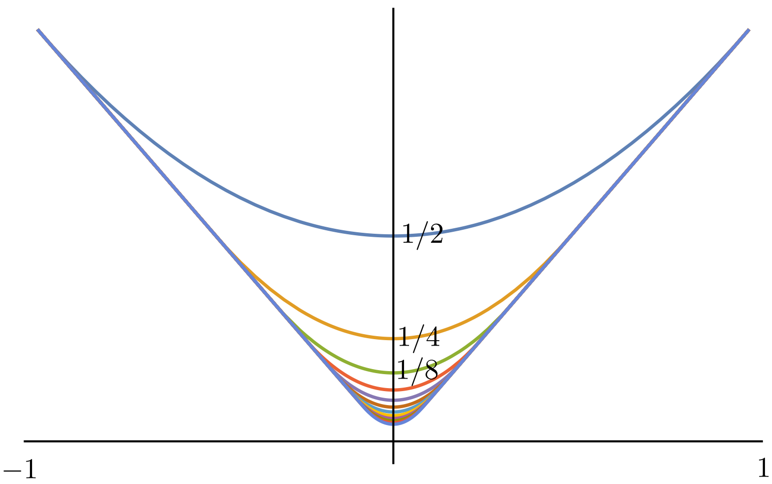

Figure 9.4: The first ten functions of the sequence $f_n \to |x|$ from Example 9.4.

$1/n$ and $f_n'(\pm 1/n) = \pm 1$. This splices the sections together at $(\pm 1/n, \pm 1/n)$ so f_n is differentiable everywhere. It's clear $f_n \to |x|$, which is not differentiable at 0.

This example shows that the limit of differentiable functions need not be differentiable.

The examples given above show that continuity, integrability and differentiability are not preserved in the pointwise limit of a sequence of functions. To have any hope of preserving these properties, a stronger form of convergence is needed.

9.2 Uniform Convergence

Definition 9.2. The sequence $f_n : S \to \mathbb{R}$ *converges uniformly* to $f : S \to \mathbb{R}$ on S, if for each $\varepsilon > 0$ there is an $N \in \mathbb{N}$ so that whenever $n \geq N$ and $x \in S$, then $|f_n(x) - f(x)| < \varepsilon$.

In this case, we write $f_n \overset{S}{\rightrightarrows} f$, or simply $f_n \rightrightarrows f$, if the set S is clear from the context.

The difference between pointwise and uniform convergence is that with pointwise convergence, the convergence of f_n to f can vary in speed at each point of S. With uniform convergence, the speed of convergence is roughly the same all across S. Uniform convergence is a stronger condition to place on the sequence f_n than pointwise convergence in the sense of the following theorem.

Theorem 9.3. *If $f_n \overset{S}{\rightrightarrows} f$, then $f_n \xrightarrow{S} f$.*

Proof. Let $x_0 \in S$ and $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that when $n \geq N$, then $|f(x) - f_n(x)| < \varepsilon$ for all $x \in S$. In particular, $|f(x_0) - f_n(x_0)| < \varepsilon$ when $n \geq N$. This shows $f_n(x_0) \to f(x_0)$. Since $x_0 \in S$ is arbitrary, it follows that $f_n \to f$. $\square$

The first three examples given above show the converse to Theorem 9.3 is false. There is, however, one interesting and useful case in which a partial converse is true.

Definition 9.4. If $f_n \xrightarrow{S} f$ and $f_n(x) \uparrow f(x)$ for all $x \in S$, then f_n *increases* to f on S. If $f_n \xrightarrow{S} f$ and $f_n(x) \downarrow f(x)$ for all $x \in S$, then f_n *decreases* to f on S. In either case, f_n is said to converge to f *monotonically*.

The functions of Example 9.4 decrease to $|x|$. Notice that in this case, the convergence is also happens to be uniform. The following theorem shows Example 9.4 to be an instance of a more general phenomenon.

Theorem 9.5 (Dini's Theorem)**.** [1] *If*

(a) *S is compact,*

(b) *$f_n \xrightarrow{S} f$ monotonically,*

(c) *$f_n \in C(S)$ for all $n \in \mathbb{N}$, and*

(d) *$f \in C(S)$,*

[1]Ulisse Dini (1845–1918) was an Italian mathematician.

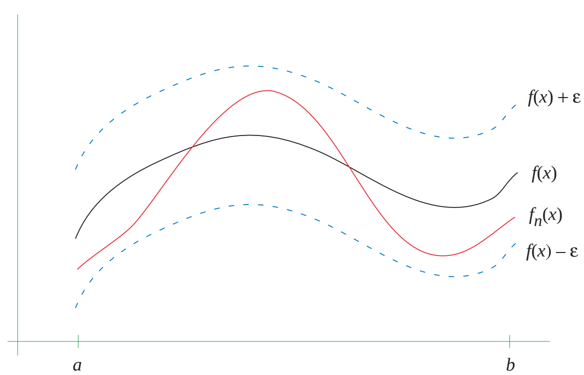

Figure 9.5: $|f_n(x) - f(x)| < \varepsilon$ on $[a, b]$, as in Definition 9.2.

then $f_n \rightrightarrows f$.

Proof. There is no loss of generality in assuming $f_n \downarrow f$, for otherwise we consider $-f_n$ and $-f$. With this assumption, if $g_n = f_n - f$, then g_n is a sequence of continuous functions decreasing to 0. It suffices to show $g_n \rightrightarrows 0$.

To do so, let $\varepsilon > 0$. Using continuity and pointwise convergence, for each $x \in S$ find an open set G_x containing x and an $N_x \in \mathbb{N}$ such that $g_{N_x}(y) < \varepsilon$ for all $y \in G_x$. Notice that the monotonicity condition guarantees $g_n(y) < \varepsilon$ for every $y \in G_x$ and $n \geq N_x$.

The collection $\{G_x : x \in S\}$ is an open cover for S, so it must contain a finite subcover $\{G_{x_i} : 1 \leq i \leq n\}$. Let $N = \max\{N_{x_i} : 1 \leq i \leq n\}$ and choose $m \geq N$. If $x \in S$, then $x \in G_{x_i}$ for some i, and $0 \leq g_m(x) \leq g_N(x) \leq g_{N_i}(x) < \varepsilon$. It follows that $g_n \rightrightarrows 0$. □

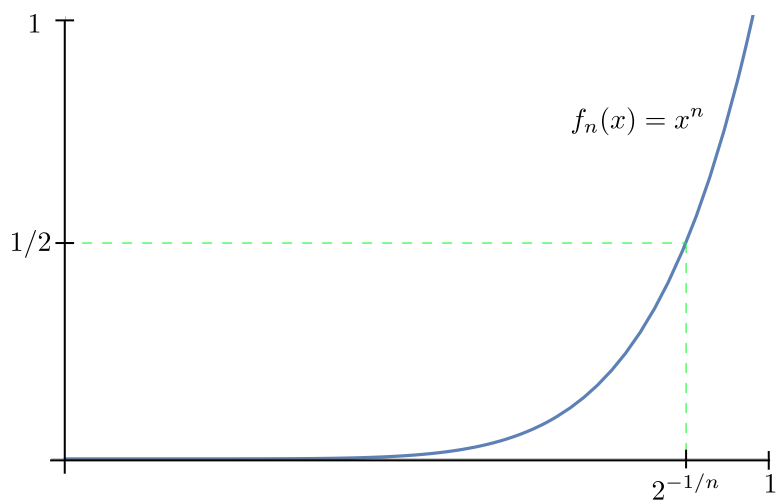

Figure 9.6: This shows a typical function from the sequence of Example 9.5.

Example 9.5. Let $f_n(x) = x^n$ for $n \in \mathbb{N}$, then f_n decreases to 0 on $[0, 1)$. If $0 < a < 1$ Dini's Theorem shows $f_n \rightrightarrows 0$ on the compact interval $[0, a]$. On the whole interval $[0, 1)$, $f_n(x) > 1/2$ when $2^{-1/n} < x < 1$, so f_n is not uniformly convergent. (Why doesn't this violate Dini's Theorem?)

9.3 Metric Properties of Uniform Convergence

If $S \subset \mathbb{R}$, let $B(S) = \{f : S \to \mathbb{R} : f \text{ is bounded}\}$. For $f \in B(S)$, define $\|f\|_S = \text{lub}\,\{|f(x)| : x \in S\}$. (It is abbreviated to $\|f\|$, if the domain S is clear from the context.) Apparently, $\|f\| \geq 0$, $\|f\| = 0 \iff f \equiv 0$ and, if $g \in B(S)$, then $\|f - g\| = \|g - f\|$. Moreover, if $h \in B(S)$, then

$$\begin{aligned}
\|f - g\| &= \text{lub}\,\{|f(x) - g(x)| : x \in S\} \\
&\leq \text{lub}\,\{|f(x) - h(x)| + |h(x) - g(x)| : x \in S\} \\
&\leq \text{lub}\,\{|f(x) - h(x)| : x \in S\} + \text{lub}\,\{|h(x) - g(x)| : x \in S\} \\
&= \|f - h\| + \|h - g\|
\end{aligned}$$

Combining all this, it follows that $\|f - g\|$ is a metric[2] on $B(S)$.

The definition of uniform convergence implies that for a sequence of bounded functions $f_n : S \to \mathbb{R}$,

$$f_n \rightrightarrows f \iff \|f_n - f\| \to 0.$$

Because of this, the metric $\|f - g\|$ is often called the *uniform metric* or the *sup-metric*. Many ideas developed using the metric properties of $\mathbb{R}$ can be carried over into this setting. In particular, there is a Cauchy criterion for uniform convergence.

Definition 9.6. Let $S \subset \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ is a *Cauchy sequence* under the uniform metric, if given $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that when $m, n \geq N$, then $\|f_n - f_m\| < \varepsilon$.

Theorem 9.7. *Let $f_n \in B(S)$. There is a function $f \in B(S)$ such that $f_n \rightrightarrows f$ iff f_n is a Cauchy sequence in $B(S)$.*

Proof. ($\Rightarrow$) Let $f_n \rightrightarrows f$ and $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that $n \geq N$ implies $\|f_n - f\| < \varepsilon/2$. If $m \geq N$ and $n \geq N$, then

$$\|f_m - f_n\| \leq \|f_m - f\| + \|f - f_n\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

shows f_n is a Cauchy sequence.

($\Leftarrow$) Suppose f_n is a Cauchy sequence in $B(S)$ and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $\|f_m - f_n\| < \varepsilon$ whenever $m \geq N$ and $n \geq N$. In particular, for a fixed $x_0 \in S$ and $m, n \geq N$, $|f_m(x_0) - f_n(x_0)| \leq \|f_m - f_n\| < \varepsilon$ shows the sequence $f_n(x_0)$ is a Cauchy sequence in $\mathbb{R}$ and therefore converges. Since x_0 is an arbitrary point of S, this defines an $f : S \to \mathbb{R}$ such that $f_n \to f$.

Finally, if $m, n \geq N$ and $x \in S$ the fact that $|f_n(x) - f_m(x)| < \varepsilon$ gives

$$|f_n(x) - f(x)| = \lim_{m \to \infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

This shows that when $n \geq N$, then $\|f_n - f\| \leq \varepsilon$. We conclude that $f \in B(S)$ and $f_n \rightrightarrows f$. $\square$

[2] Definition 2.13

A collection of functions $\mathcal{S}$ is said to be *complete* under uniform convergence, if every Cauchy sequence in $\mathcal{S}$ converges to a function in $\mathcal{S}$. Theorem 9.7 shows $B(S)$ is complete under uniform convergence. We'll see several other collections of functions that are complete under uniform convergence.

Example 9.6. For $S \subset \mathbb{R}$ let $L(S)$ be all the functions $f : S \to \mathbb{R}$ such that $f(x) = mx + b$ for some constants m and b. In particular, let f_n be a Cauchy sequence in $L([0,1])$. Theorem 9.7 shows there is an $f : [0,1] \to \mathbb{R}$ such that $f_n \rightrightarrows f$. In order to show $L([0,1])$ is complete, it suffices to show $f \in L([0,1])$.

To do this, let $f_n(x) = m_n x + b_n$ for each n. Then $f_n(0) = b_n \to f(0)$ and

$$m_n = f_n(1) - b_n \to f(1) - f(0).$$

Given any $x \in [0,1]$,

$$\begin{aligned} f_n(x) - ((f(1) - f(0))x + f(0)) &= m_n x + b_n - ((f(1) - f(0))x + f(0)) \\ &= (m_n - (f(1) - f(0)))x + b_n - f(0) \to 0. \end{aligned}$$

This shows $f(x) = (f(1) - f(0))x + f(0) \in L([0,1])$ and therefore $L([0,1])$ is complete.

Example 9.7. Let $\mathcal{P} = \{p(x) : p \text{ is a polynomial}\}$. The sequence of polynomials $p_n(x) = \sum_{k=0}^{n} x^k/k!$ increases to e^x on $[0,a]$ for any $a > 0$, so Dini's Theorem shows $p_n \rightrightarrows e^x$ on $[0,a]$. But, $e^x \notin \mathcal{P}$, so $\mathcal{P}$ is not complete. (See Exercise 25.)

9.4 Series of Functions

The definitions of pointwise and uniform convergence are extended in the natural way to series of functions. If $\sum_{k=1}^{\infty} f_k$ is a series of functions defined on a set S, then the series converges pointwise or uniformly, depending on whether the sequence of partial sums, $s_n = \sum_{k=1}^{n} f_k$ converges pointwise or uniformly, respectively. It is absolutely convergent or absolutely uniformly convergent, if $\sum_{n=1}^{\infty} |f_n|$ is convergent or uniformly convergent on S, respectively.

The following theorem is obvious and its proof is left to the reader.

Theorem 9.8. *Let $\sum_{n=1}^{\infty} f_n$ be a series of functions defined on S. If $\sum_{n=1}^{\infty} f_n$ is absolutely convergent, then it is convergent. If $\sum_{n=1}^{\infty} f_n$ is absolutely uniformly convergent, then it is uniformly convergent.*

The next theorem is a restatement of Theorem 9.5 for series.

Theorem 9.9. *If $\sum_{n=1}^{\infty} f_n$ is a series of nonnegative continuous functions converging pointwise to a continuous function on a compact set S, then $\sum_{n=1}^{\infty} f_n$ converges uniformly on S.*

A simple, but powerful technique for showing uniform convergence of series is the following.

Theorem 9.10 (Weierstrass M-Test)**.** *If $f_n : S \to \mathbb{R}$ is a sequence of functions and M_n is a sequence of numbers such that $\|f_n\|_S \le M_n$ for all $n \in \mathbb{N}$ and $\sum_{n=1}^{\infty} M_n$ converges, then $\sum_{n=1}^{\infty} f_n$ is absolutely uniformly convergent.*

Proof. Let $\varepsilon > 0$ and s_n be the sequence of partial sums of $\sum_{n=1}^{\infty} |f_n|$. Using the Cauchy criterion for convergence of a series, choose an $N \in \mathbb{N}$ such that when $n > m \ge N$, then $\sum_{k=m}^{n} M_k < \varepsilon$. So,

$$\|s_n - s_m\| = \| \sum_{k=m+1}^{n} f_k\| \le \sum_{k=m+1}^{n} \|f_k\| \le \sum_{k=m}^{n} M_k < \varepsilon.$$

This shows s_n is a Cauchy sequence and must converge according to Theorem 9.7. □

Example 9.8. Let $a > 0$ and $M_n = a^n/n!$. Since

$$\lim_{n\to\infty} \frac{M_{n+1}}{M_n} = \lim_{n\to\infty} \frac{a}{n+1} = 0,$$

the Ratio Test shows $\sum_{n=0}^{\infty} M_n$ converges. When $x \in [-a, a]$,

$$\left| \frac{x^n}{n!} \right| \le \frac{a^n}{n!}.$$

The Weierstrass M-Test now implies $\sum_{n=0}^{\infty} x^n/n!$ converges absolutely uniformly on $[-a, a]$ for any $a > 0$. (See Exercise 4.)

9.5 Continuity and Uniform Convergence

Theorem 9.11. *If $f_n : S \to \mathbb{R}$ is such that each f_n is continuous at x_0 and $f_n \overset{S}{\rightrightarrows} f$, then f is continuous at x_0.*

Proof. Let $\varepsilon > 0$. Since $f_n \rightrightarrows f$, there is an $N \in \mathbb{N}$ such that whenever $n \ge N$ and $x \in S$, then $|f_n(x) - f(x)| < \varepsilon/3$. Because f_N is continuous at x_0, there is a $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \cap S$ implies $|f_N(x) - f_N(x_0)| < \varepsilon/3$. Using these two estimates, it follows that when $x \in (x_0 - \delta, x_0 + \delta) \cap S$,

$$\begin{aligned} |f(x) - f(x_0)| &= |f(x) - f_N(x) + f_N(x) - f_N(x_0) + f_N(x_0) - f(x_0)| \\ &\le |f(x) - f_N(x)| + |f_N(x) - f_N(x_0)| + |f_N(x_0) - f(x_0)| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

Therefore, f is continuous at x_0. □

The following corollary is immediate from Theorem 9.11.

Corollary 9.12. *If f_n is a sequence of continuous functions converging uniformly to f on S, then f is continuous.*

Example 9.1 shows that continuity is not preserved under pointwise convergence. Corollary 9.12 establishes that if S is compact, then $C(S)$ is complete under the uniform metric.

The fact that $C([a,b])$ is closed under uniform convergence is often useful because, given a "bad" function $f \in C([a,b])$, it's often possible to find a sequence f_n of "good" functions in $C([a,b])$ converging uniformly to f. Following is the most widely used theorem of this type.

Theorem 9.13 (Weierstrass Approximation Theorem)**.** *If $f \in C([a,b])$, then there is a sequence of polynomials $p_n \rightrightarrows f$.*

To prove this theorem, we first need a lemma.

Lemma 9.14. *For $n \in \mathbb{N}$ let $c_n = \left(\int_{-1}^{1}(1-t^2)^n\,dt\right)^{-1}$ and*

$$k_n(t) = \begin{cases} c_n(1-t^2)^n, & |t| \le 1 \\ 0, & |t| > 1 \end{cases}.$$

(See Figure 9.7.) Then

(a) $k_n(t) \ge 0$ *for all* $t \in \mathbb{R}$ *and* $n \in \mathbb{N}$;

(b) $\int_{-1}^{1} k_n = 1$ *for all* $n \in \mathbb{N}$; *and,*

(c) *if* $0 < \delta < 1$, *then* $k_n \rightrightarrows 0$ *on* $(-\infty,-\delta] \cup [\delta,\infty)$.

Proof. Parts (a) and (b) follow easily from the definition of k_n.

To prove (c) first note that

$$1 = \int_{-1}^{1} k_n \ge \int_{-1/\sqrt{n}}^{1/\sqrt{n}} c_n(1-t^2)^n\,dt \ge c_n \frac{2}{\sqrt{n}}\left(1-\frac{1}{n}\right)^n.$$

Since $\left(1-\frac{1}{n}\right)^n \uparrow \frac{1}{e}$, it follows there is an $\alpha > 0$ such that $c_n < \alpha\sqrt{n}$.[3] Letting $\delta \in (0,1)$ and $\delta \le t \le 1$,

$$k_n(t) \le k_n(\delta) \le \alpha\sqrt{n}(1-\delta^2)^n \to 0$$

[3] Repeated application of integration by parts shows

$$c_n = \frac{n+1/2}{n} \times \frac{n-1/2}{n-1} \times \frac{n-3/2}{n-2} \times \cdots \times \frac{3/2}{1} = \frac{\Gamma(n+3/2)}{\sqrt{\pi}\,\Gamma(n+1)}.$$

With the aid of Stirling's formula, it can be shown $c_n \approx 0.565\sqrt{n} \to \infty$.

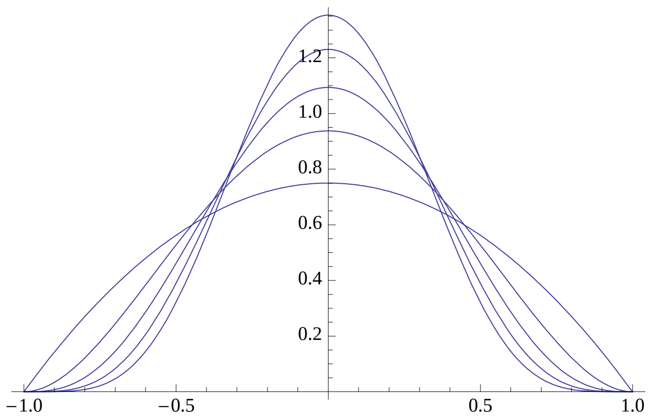

Figure 9.7: Here are the graphs of $k_n(t)$ for $n = 1, 2, 3, 4, 5$.

by L'Hospital's Rule. Since k_n is an even function, this establishes (c). □

A sequence of functions satisfying conditions such as those in Lemma 9.14 is called a *convolution kernel* or a *Dirac sequence*.[4] Several such kernels play a key role in the study of Fourier series, as we will see in Theorems 10.5 and 10.13. The one defined above is called the Landau kernel.[5]

We now turn to the proof of the theorem.

Proof. There is no generality lost in assuming $[a, b] = [0, 1]$, for otherwise we consider the linear change of variables $g(x) = f((b - a)x + a)$. Similarly, we can assume $f(0) = f(1) = 0$, for otherwise we consider $g(x) = f(x) - ((f(1) - f(0))x - f(0)$, which is a polynomial added to f. We can further assume $f(x) = 0$ when $x \notin [0, 1]$.

Set

$$p_n(x) = \int_{-1}^{1} f(x + t)k_n(t)\, dt. \tag{9.1}$$

[4]Given two functions f and g defined on $\mathbb{R}$, the *convolution* of f and g is the integral

$$f \star g(x) = \int_{-\infty}^{\infty} f(t)g(x - t)\, dt.$$

The term convolution kernel is used because such kernels typically replace g in the convolution given above, as can be seen in the proof of the Weierstrass approximation theorem.

[5]It was investigated by the German mathematician Edmund Landau (1877–1938).

To see p_n is a polynomial, change variables in the integral using $u = x + t$ to arrive at

$$p_n(x) = \int_{x-1}^{x+1} f(u)k_n(u-x)\,du = \int_0^1 f(u)k_n(x-u)\,du,$$

because $f(x) = 0$ when $x \notin [0,1]$. Notice that $k_n(x-u)$ is a polynomial in u with coefficients being polynomials in x, so integrating $f(u)k_n(x-u)$ yields a polynomial in x. (Just try it for a small value of n and a simple function f!)

Use (9.1) and Lemma 9.14(b) to see for $\delta \in (0,1)$ that

$$\begin{aligned}
|p_n(x) - f(x)| &= \left| \int_{-1}^{1} f(x+t)k_n(t)\,dt - f(x) \right| \\
&= \left| \int_{-1}^{1} (f(x+t) - f(x))k_n(t)\,dt \right| \\
&\le \int_{-1}^{1} |f(x+t) - f(x)|k_n(t)\,dt \\
&= \int_{-\delta}^{\delta} |f(x+t) - f(x)|k_n(t)\,dt + \int_{\delta<|t|\le 1} |f(x+t) - f(x)|k_n(t)\,dt. \qquad (9.2)
\end{aligned}$$

We'll handle each of the final integrals in turn.

Let $\varepsilon > 0$ and use the uniform continuity of f to choose a $\delta \in (0,1)$ such that when $|t| < \delta$, then $|f(x+t) - f(x)| < \varepsilon/2$. Then, using Lemma 9.14(b) again,

$$\int_{-\delta}^{\delta} |f(x+t) - f(x)|k_n(t)\,dt < \frac{\varepsilon}{2}\int_{-\delta}^{\delta} k_n(t)\,dt < \frac{\varepsilon}{2}. \qquad (9.3)$$

According to Lemma 9.14(c), there is an $N \in \mathbb{N}$ so that when $n \ge N$ and $|t| \ge \delta$, then $k_n(t) < \frac{\varepsilon}{8(\|f\|+1)(1-\delta)}$. Using this, it follows that

$$\begin{aligned}
&\int_{\delta<|t|\le 1} |f(x+t) - f(x)|k_n(t)\,dt \\
&\quad = \int_{-1}^{-\delta} |f(x+t) - f(x)|k_n(t)\,dt + \int_{\delta}^{1} |f(x+t) - f(x)|k_n(t)\,dt \\
&\quad \le 2\|f\| \int_{-1}^{-\delta} k_n(t)\,dt + 2\|f\| \int_{\delta}^{1} k_n(t)\,dt \\
&\quad < 2\|f\| \frac{\varepsilon}{8(\|f\|+1)(1-\delta)}(1-\delta) + 2\|f\| \frac{\varepsilon}{8(\|f\|+1)(1-\delta)}(1-\delta) = \frac{\varepsilon}{2} \qquad (9.4)
\end{aligned}$$

Combining (9.3) and (9.4), it follows from (9.2) that $|p_n(x) - f(x)| < \varepsilon$ for all $x \in [0,1]$ and $p_n \rightrightarrows f$. □

Corollary 9.15. *If $f \in C([a,b])$ and $\varepsilon > 0$, then there is a polynomial p such that $\|f - p\|_{[a,b]} < \varepsilon$.*

The theorems of this section can also be used to construct some striking examples of functions with unwelcome behavior. Following is perhaps the most famous.

Example 9.9. There is a continuous $f : \mathbb{R} \to \mathbb{R}$ that is differentiable nowhere.

Proof. Thinking of the canonical example of a continuous function that fails to be differentiable at a point—the absolute value function—we start with a "sawtooth" function. (See Figure 9.8.)

$$s_0(x) = \begin{cases} x - 2n, & 2n \le x < 2n+1,\ n \in \mathbb{Z} \\ 2n + 2 - x, & 2n+1 \le x < 2n+2,\ n \in \mathbb{Z} \end{cases}$$

Notice that s_0 is continuous and periodic with period 2 and maximum value 1. Compress it both vertically and horizontally:

$$s_n(x) = \left(\frac{3}{4}\right)^n s_0\left(4^n x\right),\ n \in \mathbb{N}.$$

Each s_n is continuous and periodic with period $p_n = 2/4^n$ and $\|s_n\| = (3/4)^n$.

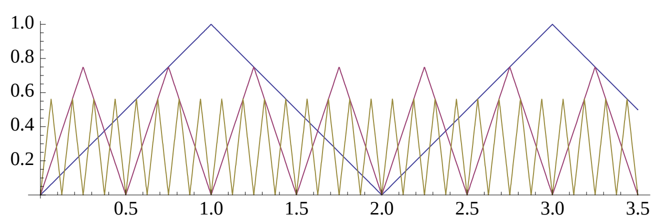

Figure 9.8: s_0, s_1 and s_2 from Example 9.9.

Finally, the desired function is

$$f(x) = \sum_{n=0}^{\infty} s_n(x).$$

Since $\|s_n\| = (3/4)^n$, the Weierstrass M-test implies the series defining f is uniformly convergent and Corollary 9.12 shows f is continuous on $\mathbb{R}$. We will show f is differentiable nowhere.

Let $x \in \mathbb{R}$, $m \in \mathbb{N}$ and $h_m = 1/(2 \cdot 4^m)$.

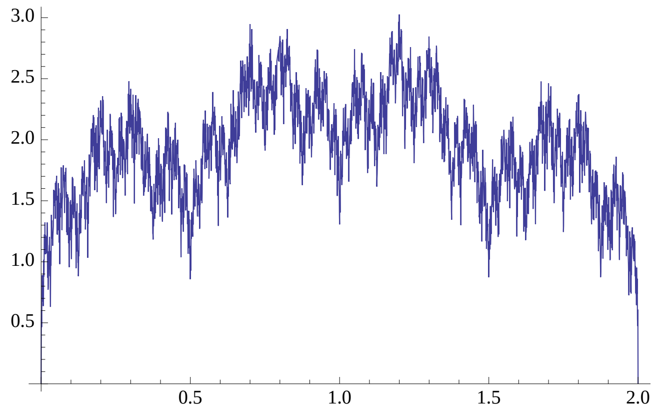

Figure 9.9: The nowhere differentiable function f from Example 9.9. It is periodic with period 2 and one complete period is shown.

If $n > m$, then $h_m / p_n = 4^{n-m-1} \in \omega$, so $s_n(x \pm h_m) - s_n(x) = 0$ and

$$\frac{f(x \pm h_m) - f(x)}{\pm h_m} = \sum_{k=0}^{m} \frac{s_k(x \pm h_m) - s_k(x)}{\pm h_m}. \tag{9.5}$$

On the other hand, if $n < m$, then a worst-case estimate is

$$\left| \frac{s_n(x \pm h_m) - s_n(x)}{h_m} \right| \leq \left(\frac{3}{4} \right)^n / \left(\frac{1}{4^n} \right) = 3^n.$$

This gives

$$\begin{aligned} \left| \sum_{k=0}^{m-1} \frac{s_k(x \pm h_m) - s_k(x)}{\pm h_m} \right| &\leq \sum_{k=0}^{m-1} \left| \frac{s_k(x \pm h_m) - s_k(x)}{\pm h_m} \right| \\ &\leq \frac{3^m - 1}{3 - 1} < \frac{3^m}{2}. \end{aligned} \tag{9.6}$$

Since s_m is linear on intervals of length $4^{-m} = 2 \cdot h_m$ with slope $\pm 3^m$ on those linear segments, at least one of the following is true:

$$\left| \frac{s_m(x + h_m) - s(x)}{h_m} \right| = 3^m \text{ or } \left| \frac{s_m(x - h_m) - s(x)}{-h_m} \right| = 3^m. \tag{9.7}$$

Suppose the first of these is true. The argument is essentially the same in the second case.

Using (9.5), (9.6) and (9.7), the following estimate ensues

$$\begin{aligned}
\left|\frac{f(x+h_m)-f(x)}{h_m}\right| &= \left|\sum_{k=0}^{\infty}\frac{s_k(x+h_m)-s_k(x)}{h_m}\right| \\
&= \left|\sum_{k=0}^{m}\frac{s_k(x+h_m)-s_k(x)}{h_m}\right| \\
&\geq \left|\frac{s_m(x+h_m)-s_m(x)}{h_m}\right| - \sum_{k=0}^{m-1}\left|\frac{s_k(x+h_m)-s_k(x)}{h_m}\right| \\
&> 3^m - \frac{3^m}{2} = \frac{3^m}{2}.
\end{aligned}$$

Since $3^m/2 \to \infty$, it is apparent $f'(x)$ does not exist. □

There are many other constructions of nowhere differentiable continuous functions. The first was published by Weierstrass [22] in 1872, although it was known in the folklore sense among mathematicians earlier than this. (There is an English translation of Weierstrass' paper in [12].) In fact, it is now known in a technical sense that the "typical" continuous function is nowhere differentiable [5].

9.6 Integration and Uniform Convergence

One of the recurring questions with integrals is when it is true that

$$\lim_{n\to\infty}\int f_n = \int \lim_{n\to\infty} f_n.$$

This is often referred to as "passing the limit through the integral." At some point in her career, any student of advanced analysis or probability theory will be tempted to just blithely pass the limit through. But functions such as those of Example 9.3 show that some care is needed. A common criterion for doing so is uniform convergence.

Theorem 9.16. *If $f_n : [a,b] \to \mathbb{R}$ such that $\int_a^b f_n$ exists for each n and $f_n \rightrightarrows f$ on $[a,b]$, then*

$$\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$$

.

Proof. Some care must be taken in this proof, because there are actually two things to prove. Before the equality can be shown, it must be proved that f is integrable.

To show that f is integrable, let $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $\|f - f_N\| < \varepsilon/3(b-a)$. If $P \in \operatorname{part}([a,b])$, then

$$\begin{aligned} |\mathcal{R}(f, P, x_k^*) - \mathcal{R}(f_N, P, x_k^*)| &= |\sum_{k=1}^{n} f(x_k^*)|I_k| - \sum_{k=1}^{n} f_N(x_k^*)|I_k|| \\ &= |\sum_{k=1}^{n}(f(x_k^*) - f_N(x_k^*))|I_k|| \\ &\leq \sum_{k=1}^{n}|f(x_k^*) - f_N(x_k^*)||I_k| \\ &< \frac{\varepsilon}{3(b-a))}\sum_{k=1}^{n}|I_k| \\ &= \frac{\varepsilon}{3} \end{aligned} \tag{9.8}$$

According to Theorem 8.10, there is a $P \in \operatorname{part}([a,b])$ such that whenever $P \ll Q_1$ and $P \ll Q_2$, then

$$|\mathcal{R}(f_N, Q_1, x_k^*) - \mathcal{R}(f_N, Q_2, y_k^*)| < \frac{\varepsilon}{3}. \tag{9.9}$$

Combining (9.8) and (9.9) yields

$$\begin{aligned} &|\mathcal{R}(f, Q_1, x_k^*) - \mathcal{R}(f, Q_2, y_k^*)| \\ &\quad = |\mathcal{R}(f, Q_1, x_k^*) - \mathcal{R}(f_N, Q_1, x_k^*) + \mathcal{R}(f_N, Q_1, x_k^*) \\ &\qquad - \mathcal{R}(f_N, Q_2, y_k^*) + \mathcal{R}(f_N, Q_2, y_k^*) - \mathcal{R}(f, Q_2, y_k^*)| \\ &\leq |\mathcal{R}(f, Q_1, x_k^*) - \mathcal{R}(f_N, Q_1, x_k^*)| + |\mathcal{R}(f_N, Q_1, x_k^*) - \mathcal{R}(f_N, Q_2, y_k^*)| \\ &\qquad + |\mathcal{R}(f_N, Q_2, y_k^*) - \mathcal{R}(f, Q_2, y_k^*)| \\ &\qquad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \end{aligned}$$

Another application of Theorem 8.10 shows that f is integrable.

Finally, when $n \geq N$,

$$\left|\int_a^b f - \int_a^b f_n\right| = \left|\int_a^b (f - f_n)\right| < \int_a^b \frac{\varepsilon}{3(b-a)} = \frac{\varepsilon}{3} < \varepsilon$$

shows that $\int_a^b f_n \to \int_a^b f$. □

Corollary 9.17. *If $\sum_{n=1}^{\infty} f_n$ is a series of integrable functions converging uniformly on $[a, b]$, then*

$$\int_a^b \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_a^b f_n$$

Use of this corollary is sometimes referred to as "reversing summation and integration." It's tempting to do this reversal, but without some condition such as uniform convergence, justification for the action is often difficult.

Example 9.10. It was shown in Example 4.2 that the geometric series

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}, \quad -1 < t < 1.$$

In Exercise 3, you are asked to prove this convergence is uniform on any compact subset of $(-1, 1)$. Substituting $-t$ for t in the above formula, it follows that

$$\sum_{n=0}^{\infty} (-t)^n \rightrightarrows \frac{1}{1+t}$$

on $[0, x]$, when $0 < x < 1$. Corollary 9.17 implies

$$\ln(1 + x) = \int_0^x \frac{dt}{1+t} = \sum_{n=0}^{\infty} \int_0^x (-t)^n \, dt = x - x^2 + x^3 - x^4 + \cdots .$$

The same argument works when $-1 < x < 0$, so

$$\ln(1 + x) = x - x^2 + x^3 - x^4 + \cdots$$

when $x \in (-1, 1)$.

Combining Theorem 9.16 with Dini's Theorem, gives the following.

Corollary 9.18 (Monotone Convergence Theorem)**.** *If f_n is a sequence of continuous functions converging monotonically to a continuous function f on $[a, b]$, then $\int_a^b f_n \to \int_a^b f$.*

9.7 Differentiation and Uniform Convergence

The relationship between uniform convergence and differentiation is somewhat more complex than those relationships we've already examined. First, because there are two sequences involved, f_n and f_n', either of which may converge or diverge at a point; and second, because differentiation is more "delicate" than continuity or integration.

Example 9.4 is an explicit example of a sequence of differentiable functions converging uniformly to a function which is not differentiable at a point. The derivatives of the functions from that example converge pointwise to a function that is not a derivative. Combining the Weierstrass Approximation Theorem (9.13) and Example 9.9 pushes this to the extreme by showing the existence of a sequence of polynomials converging uniformly to a continuous nowhere differentiable function.

The following theorem starts to shed some light on the situation.

Theorem 9.19. *If f_n is a sequence of derivatives defined on $[a,b]$ and $f_n \rightrightarrows f$, then f is a derivative.*

Proof. For each n, let F_n be an antiderivative of f_n. By considering $F_n(x) - F_n(a)$, if necessary, there is no generality lost with the assumption that $F_n(a) = 0$ for all n.

Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ such that

$$m, n \geq N \implies \|f_m - f_n\| < \frac{\varepsilon}{b-a}.$$

If $x \in [a,b]$ and $m, n \geq N$, then the Mean Value Theorem and the assumption that $F_m(a) = F_n(a) = 0$ yield a $c \in [a,x)$ such that

$$\begin{aligned}|F_m(x) - F_n(x)| &= |(F_m(x) - F_n(x)) - (F_m(a) - F_n(a))| \\ &= |f_m(c) - f_n(c)|\,|x-a| \leq \|f_m - f_n\|(b-a) < \varepsilon. \end{aligned} \tag{9.10}$$

This shows F_n is a Cauchy sequence in $C([a,b])$ and there is an $F \in C([a,b])$ with $F_n \rightrightarrows F$.

It suffices to show $F' = f$. To do this, several estimates are established.

Let $N \in \mathbb{N}$ so that

$$m, n \geq N \implies \|f_m - f_n\| < \frac{\varepsilon}{3}.$$

Notice this implies

$$\|f - f_n\| \leq \frac{\varepsilon}{3},\ \forall n \geq M. \tag{9.11}$$

For such $m, n \geq N$ and $x, y \in [a, b]$ with $x \neq y$, another application of the Mean Value Theorem gives

$$\begin{aligned}\left|\frac{F_n(x) - F_n(y)}{x - y} - \frac{F_m(x) - F_m(y)}{x - y}\right| &\\ &= \frac{1}{|x-y|}\left|(F_n(x) - F_m(x)) - (F_n(y) - F_m(y))\right| \\ &= \frac{1}{|x-y|}\left|f_n(c) - f_m(c)\right||x - y| \leq \|f_n - f_m\| < \frac{\varepsilon}{3}.\end{aligned}$$

Letting $m \to \infty$, it follows that

$$\left|\frac{F_n(x) - F_n(y)}{x - y} - \frac{F(x) - F(y)}{x - y}\right| \leq \frac{\varepsilon}{3},\ \forall n \geq M. \tag{9.12}$$

Fix $n \geq N$ and $x \in [a, b]$. Since $F_n'(x) = f_n(x)$, there is a $\delta > 0$ so that

$$\left|\frac{F_n(x) - F_n(y)}{x - y} - f_n(x)\right| < \frac{\varepsilon}{3}, \forall y \in (x - \delta, x + \delta) \setminus \{x\}. \tag{9.13}$$

Finally, using (9.12), (9.13) and (9.11), we see

$$\begin{aligned}\left|\frac{F(x) - F(y)}{x - y} - f(x)\right| &\\ &= \left|\frac{F(x) - F(y)}{x - y} - \frac{F_n(x) - F_n(y)}{x - y}\right. \\ &\qquad \left. + \frac{F_n(x) - F_n(y)}{x - y} - f_n(x) + f_n(x) - f(x)\right| \\ &\leq \left|\frac{F(x) - F(y)}{x - y} - \frac{F_n(x) - F_n(y)}{x - y}\right| \\ &\qquad + \left|\frac{F_n(x) - F_n(y)}{x - y} - f_n(x)\right| + |f_n(x) - f(x)| \\ &< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

This establishes that

$$\lim_{y \to x} \frac{F(x) - F(y)}{x - y} = f(x),$$

as desired. □

Corollary 9.20. *If $G_n \in C([a, b])$ is a sequence such that $G_n' \rightrightarrows g$ and $G_n(x_0)$ converges for some $x_0 \in [a, b]$, then $G_n \rightrightarrows G$ where $G' = g$.*

Proof. Let $G_n' = g_n$ and let $G_n(x_0) \to \alpha$. For each n choose an antiderivative F_n of g_n such that $F_n(a) = 0$. Theorem 9.19 shows g is a derivative and an argument similar to that in the proof of Theorem 9.19 shows $F_n \rightrightarrows F$ on $[a, b]$, where $F' = g$. Since $F_n' - G_n' = 0$, Corollary (7.16) shows $G_n(x) = F_n(x) + (G_n(x_0) - F_n(x_0))$. Define $G(x) = F(x) + (\alpha - F(x_0))$.

Let $\varepsilon > 0$ and $x \in [a, b]$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \implies \|F_n - F\| < \frac{\varepsilon}{3} \text{ and } |G_n(x_0) - \alpha| < \frac{\varepsilon}{3}.$$

If $n \geq N$,

$$\begin{aligned}
|G_n(x) - G(x)| &= |F_n(x) + (G_n(x_0) - F_n(x_0)) - (F(x) + (\alpha - F(x_0)))| \\
&\leq |F_n(x) - F(x)| + |G_n(x_0) - \alpha| + |F_n(x_0) - F(x_0)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

This shows $G_n \rightrightarrows G$ on $[a, b]$ where $G' = F' = g$. $\square$

Corollary 9.21. *If f_n is a sequence of differentiable functions defined on $[a, b]$ such that $\sum_{k=1}^{\infty} f_k(x_0)$ exists for some $x_0 \in [a, b]$ and $\sum_{k=1}^{\infty} f_k'$ converges uniformly, then*

$$\left(\sum_{k=1}^{\infty} f_k\right)' = \sum_{k=1}^{\infty} f_k'$$

Proof. Left as an exercise. $\square$

Example 9.11. Let $a > 0$ and $f_n(x) = x^n/n!$. Note that $f_n' = f_{n-1}$ for $n \in \mathbb{N}$. Example 9.8 shows $\sum_{n=0}^{\infty} f_n'(x)$ is uniformly convergent on $[-a, a]$. Corollary 9.21 shows

$$\left(\sum_{n=0}^{\infty} \frac{x^n}{n!}\right)' = \sum_{n=0}^{\infty} \frac{x^n}{n!}. \tag{9.14}$$

on $[-a, a]$. Since a is an arbitrary positive constant, (9.14) is seen to hold on all of $\mathbb{R}$.

If $f(x) = \sum_{n=0}^{\infty} x^n/n!$, then the argument given above implies the initial value problem

$$\begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$$

As is well-known, the unique solution to this problem is $f(x) = e^x$. Therefore,

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

9.8 Power Series

9.8.1 The Radius and Interval of Convergence

One place where uniform convergence plays a key role is with power series. Recall the definition.

Definition 9.22. A *power series* is a function of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n. \tag{9.15}$$

Members of the sequence a_n are the *coefficients* of the series. The domain of f is the set of all x at which the series converges. The constant c is called the *center* of the series.

To determine the domain of (9.15), let $x \in \mathbb{R} \setminus \{c\}$ and use the Root Test to see the series converges when

$$\limsup |a_n(x-c)^n|^{1/n} = |x-c| \limsup |a_n|^{1/n} < 1$$

and diverges when

$$|x-c| \limsup |a_n|^{1/n} > 1.$$

If $r \limsup |a_n|^{1/n} \leq 1$ for some $r \geq 0$, then these inequalities imply (9.15) is absolutely convergent when $|x-c| < r$. In other words, if

$$R = \text{lub}\,\{r : r \limsup |a_n|^{1/n} < 1\}, \tag{9.16}$$

then the domain of (9.15) is an interval of radius R centered at c. The root test gives no information about convergence when $|x-c| = R$. This R is called the *radius of convergence* of the power series. Assuming $R > 0$, the open interval centered at c with radius R is called the *interval of convergence.* It may be different from the domain of the series because the series may converge at neither, one, or both endpoints of the interval of convergence. [6]

The ratio test can also be used to determine the radius of convergence, but, as shown in (4.9), it will not work as often as the root test. When it does,

$$R = \text{lub}\,\left\{r : r \limsup \left|\frac{a_{n+1}}{a_n}\right| < 1\right\}. \tag{9.17}$$

This is usually easier to compute than (9.16), and both will give the same value for R, when they can both be evaluated.

[6] Some authors define the interval of convergence to be the same as what we call the domain.

In practice, the radius of convergence can most often determined by considering

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} \text{ or } \frac{1}{R} = \lim_{n\to\infty} \sqrt[n]{|a_n|}.$$

The first of these limits is usually easier to calculate.

Example 9.12. Calling to mind Example 4.2, it is apparent the geometric power series $\sum_{n=0}^{\infty} x^n$ has center 0, radius of convergence 1 and domain $(-1, 1)$.

Example 9.13. For the power series $\sum_{n=1}^{\infty} 2^n (x+2)^n / n$, we compute

$$\limsup \left(\frac{2^n}{n}\right)^{1/n} = 2 \implies R = \frac{1}{2}.$$

Since the series diverges when $x = -2 \pm \frac{1}{2}$, it follows that the domain is the interval of convergence $(-5/2, -3/2)$.

Example 9.14. The power series $\sum_{n=1}^{\infty} x^n / n$ has interval of convergence $(-1, 1)$ and domain $[-1, 1)$. Notice it is not absolutely convergent when $x = -1$.

Example 9.15. The power series $\sum_{n=1}^{\infty} x^n / n^2$ has interval of convergence $(-1, 1)$, domain $[-1, 1]$ and is absolutely convergent on its whole domain.

The preceding is summarized in the following theorem.

Theorem 9.23. *Let the power series be as in (9.15) and R be given by either (9.16) or (9.17).*

(a) *If $R = 0$, then the domain of the series is $\{c\}$.*

(b) *If $R > 0$ the series converges absolutely at x when $|c - x| < R$ and diverges at x when $|c - x| > R$. In the case when $R = \infty$, the series converges everywhere.*

(c) *If $R \in (0, \infty)$, then the series may converge at none, one or both of $c - R$ and $c + R$.*

9.8.2 Uniform Convergence of Power Series

The partial sums of a power series are a sequence of polynomials converging pointwise on the domain of the series. As has been seen, pointwise convergence is not enough to say much about the behavior of the power series. The following theorem opens the door to a lot more.

Theorem 9.24. *A power series converges absolutely and uniformly on compact subsets of its interval of convergence.*

Proof. There is no generality lost in assuming the series has the form of (9.15) with $c = 0$. Let the radius of convergence be $R > 0$ and K be a compact subset of $(-R, R)$ with $\alpha = \text{lub}\,\{|x| : x \in K\}$. Choose $r \in (\alpha, R)$. If $x \in K$, then $|a_n x^n| < |a_n r^n|$ for $n \in \mathbb{N}$. Since $\sum_{n=0}^{\infty} |a_n r^n|$ converges, the Weierstrass M-test shows $\sum_{n=0}^{\infty} a_n x^n$ is absolutely and uniformly convergent on K. □

The following two corollaries are immediate consequences of Corollary 9.12 and Theorem 9.16, respectively.

Corollary 9.25. *A power series is continuous on its interval of convergence.*

Corollary 9.26. *If $[a, b]$ is an interval contained in the interval of convergence for the power series $\sum_{n=0}^{\infty} a_n (x-c)^n$, then*

$$\int_a^b \sum_{n=0}^{\infty} a_n (x-c)^n = \sum_{n=0}^{\infty} a_n \int_a^b (x-c)^n.$$

Example 9.16. Define[7]

$$f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}.$$

Since $\lim_{x \to 0} f(x) = 1$, f is continuous everywhere. Suppose we want $\int_0^\pi f$ with an accuracy of five decimal places.

If $x \neq 0$,

$$f(x) = \frac{1}{x} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)!} x^{2n-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}$$

The latter series converges to f everywhere. Corollary 9.26 implies

$$\begin{aligned} \int_0^\pi f(x)\,dx &= \int_0^\pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n}\,dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int_0^\pi x^{2n}\,dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(2n+1)!} \pi^{2n+1} \qquad (9.18) \end{aligned}$$

The latter series satisfies the Alternating Series Test. Since $\pi^{15}/(15 \times 15!) \approx 1.5 \times 10^{-6}$, Corollary 4.21 shows

$$\int_0^\pi f(x)\,dx \approx \sum_{n=0}^{6} \frac{(-1)^n}{(2n+1)(2n+1)!} \pi^{2n+1} \approx 1.85194$$

[7]This is often called the *sinc* function. (It sounds the same as "sink.")

The next question is: What about differentiability?

Notice that the continuity of the exponential function and L'Hospital's Rule give

$$\lim_{n\to\infty} n^{1/n} = \lim_{n\to\infty} \exp\left(\frac{\ln n}{n}\right) = \exp\left(\lim_{n\to\infty}\frac{\ln n}{n}\right) = \exp(0) = 1.$$

Therefore, for any sequence a_n,

$$\limsup |na_n|^{1/n} = \limsup n^{1/n}|a_n|^{1/n} = \limsup |a_n|^{1/n}. \tag{9.19}$$

Now, suppose the power series $\sum_{n=0}^{\infty} a_n x^n$ has a nontrivial interval of convergence, I. Formally differentiating the power series term-by-term gives a new power series $\sum_{n=1}^{\infty} na_n x^{n-1}$. According to (9.19) and Theorem 9.23, the term-by-term differentiated series has the same interval of convergence as the original. Its partial sums are the derivatives of the partial sums of the original series and Theorem 9.24 guarantees they converge uniformly on any compact subset of I. Corollary 9.21 shows

$$\frac{d}{dx}\sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty}\frac{d}{dx}a_n x^n = \sum_{n=1}^{\infty} na_n x^{n-1},\ \forall x \in I.$$

This process can be continued inductively to obtain the same results for all higher order derivatives. We have proved the following theorem.

Theorem 9.27. *If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is a power series with nontrivial interval of convergence, I, then f is differentiable to all orders on I with*

$$f^{(m)}(x) = \sum_{n=m}^{\infty}\frac{n!}{(n-m)!}a_n(x-c)^{n-m}. \tag{9.20}$$

Moreover, the differentiated series has I as its interval of convergence.

9.8.3 Taylor Series

Suppose $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has $I = (-R, R)$ as its interval of convergence for some $R > 0$. According to Theorem 9.27,

$$f^{(m)}(0) = \frac{m!}{(m-m)!}a_m \implies a_m = \frac{f^{(m)}(0)}{m!},\ \forall m \in \omega.$$

Therefore,

$$f(x) = \sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n,\ \forall x \in I.$$

This is a remarkable result! It shows that the values of f on I are completely determined by its values on any neighborhood of 0. This is summarized in the following theorem.

Theorem 9.28. *If a power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ has a nontrivial interval of convergence I, then*

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n, \ \forall x \in I. \tag{9.21}$$

The series (9.21) is called the *Taylor series*[8] for f centered at c. The Taylor series can be formally defined for any function that has derivatives of all orders at c, but, as Example 7.9 shows, there is no guarantee it will converge to the function anywhere except at c. Taylor's Theorem 7.18 can be used to examine the question of pointwise convergence. If f can be represented by a power series on an open interval I, then f is said to be *analytic* on I.

9.8.4 The Endpoints of the Interval of Convergence

We have seen that at the endpoints of its interval of convergence a power series may diverge or even absolutely converge. A natural question when it does converge is the following: What is the relationship between the value at the endpoint and the values inside the interval of convergence?

Theorem 9.29 (Abel)**.** *A power series is continuous on its domain.*

Proof. Let $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ have interval of convergence I. If $I = \{c\}$, the theorem is vacuously true from Definition 6.9. If $I = \mathbb{R}$, the theorem follows from Corollary 9.25. So, assume its interval of convergence is $R \in (0, \infty)$. It must be shown that if f converges at an endpoint of $I = (c-R, c+R)$, then f is continuous at that endpoint.

It can be assumed $c = 0$ and $R = 1$. There is no loss of generality with either of these assumptions because otherwise just replace $f(x)$ with $f((x+c)/R)$. The theorem will be proved for $\alpha = 1$ since the other case is proved similarly.

[8] When $c = 0$, it is often called the *Maclaurin series* for f, after the Scottish mathematician Colin Maclaurin (1698–1746).

Set $s = f(1)$, $s_{-1} = 0$ and $s_n = \sum_{k=0}^{n} a_k$ for $n \in \omega$. For $|x| < 1$,

$$\begin{aligned}
\sum_{k=0}^{n} a_k x^k &= \sum_{k=0}^{n} (s_k - s_{k-1}) x^k \\
&= \sum_{k=0}^{n} s_k x^k - \sum_{k=1}^{n} s_{k-1} x^k \\
&= s_n x^n + \sum_{k=0}^{n-1} s_k x^k - x \sum_{k=0}^{n-1} s_k x^k \\
&= s_n x^n + (1-x) \sum_{k=0}^{n-1} s_k x^k
\end{aligned}$$

When $n \to \infty$, since s_n is bounded and $|x| < 1$,

$$f(x) = (1-x) \sum_{k=0}^{\infty} s_k x^k. \tag{9.22}$$

Since $(1-x)\sum_{n=0}^{\infty} x^n = 1$, (9.22) implies

$$|f(x) - s| = \left| (1-x) \sum_{k=0}^{\infty} (s_k - s) x^k \right|. \tag{9.23}$$

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ such that whenever $n \geq N$, then $|s_n - s| < \varepsilon/2$. Choose $\delta \in (0,1)$ so

$$\delta \sum_{k=0}^{N} |s_k - s| < \varepsilon/2.$$

Suppose x is such that $1 - \delta < x < 1$. With these choices, (9.23) becomes

$$\begin{aligned}
|f(x) - s| &\leq \left| (1-x) \sum_{k=0}^{N} (s_k - s) x^k \right| + \left| (1-x) \sum_{k=N+1}^{\infty} (s_k - s) x^k \right| \\
&< \delta \sum_{k=0}^{N} |s_k - s| + \frac{\varepsilon}{2} \left| (1-x) \sum_{k=N+1}^{\infty} x^k \right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}$$

It has been shown that $\lim_{x \uparrow 1} f(x) = f(1)$, so $1 \in C(f)$. □

Here is an example showing the power of these techniques.

Example 9.17. The series

$$\sum_{n=0}^{\infty}(-1)^n x^{2n} = \frac{1}{1+x^2}$$

has $(-1,1)$ as its interval of convergence. If $0 \le |x| < 1$, then Corollary 9.17 justifies

$$\arctan(x) = \int_0^x \frac{dt}{1+t^2} = \int_0^x \sum_{n=0}^{\infty}(-1)^n t^{2n}\,dt = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}.$$

This series for the arctangent converges by the alternating series test when $x = 1$, so Theorem 9.29 implies

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1} = \lim_{x\uparrow 1}\arctan(x) = \arctan(1) = \frac{\pi}{4}. \tag{9.24}$$

A bit of rearranging gives the formula

$$\pi = 4\left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right),$$

which is known as Gregory's[9] series for π.

Finally, Abel's theorem opens up an interesting idea for the summation of series. Suppose $\sum_{n=0}^{\infty} a_n$ is a series. The *Abel sum* of this series is

$$A\sum_{n=0}^{\infty} a_n = \lim_{x\uparrow 1}\sum_{n=0}^{\infty} a_n x^n.$$

Consider the following example.

Example 9.18. Let $a_n = (-1)^n$ so

$$\sum_{n=0}^{\infty} a_n = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$$

diverges. But,

$$A\sum_{n=0}^{\infty} a_n = \lim_{x\uparrow 1}\sum_{n=0}^{\infty}(-x)^n = \lim_{x\uparrow 1}\frac{1}{1+x} = \frac{1}{2}.$$

[9] James Gregory (1638–1675) was likely the first to publish a proof of the fundamental theorem of calculus in 1667.

This shows the Abel sum of a series may exist when the ordinary sum does not. Abel's theorem guarantees when both exist they are the same.

Abel summation is one of many different summation methods used in areas such as harmonic analysis. (For another see Exercise 28.)

Theorem 9.30 (Tauber)**.** *If $\sum_{n=0}^{\infty} a_n$ is a series satisfying*

(a) $na_n \to 0$ *and*

(b) $A\sum_{n=0}^{\infty} a_n = A$,

then $\sum_{n=0}^{\infty} a_n = A$.

Proof. Let $s_n = \sum_{k=0}^{n} a_k$. For $x \in (0,1)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
\left| s_n - \sum_{k=0}^{\infty} a_k x^k \right| &= \left| \sum_{k=0}^{n} a_k - \sum_{k=0}^{n} a_k x^k - \sum_{k=n+1}^{\infty} a_k x^k \right| \\
&= \left| \sum_{k=0}^{n} a_k (1 - x^k) - \sum_{k=n+1}^{\infty} a_k x^k \right| \\
&= \left| \sum_{k=0}^{n} a_k (1 - x)(1 + x + \cdots + x^{k-1}) - \sum_{k=n+1}^{\infty} a_k x^k \right| \\
&\le (1 - x) \sum_{k=0}^{n} k|a_k| + \sum_{k=n+1}^{\infty} |a_k| x^k. \qquad (9.25)
\end{aligned}$$

Let $\varepsilon > 0$. According to (a) and Exercise 27, there is an $N \in \mathbb{N}$ such that

$$n \ge N \implies n|a_n| < \frac{\varepsilon}{2} \text{ and } \frac{1}{n} \sum_{k=0}^{n} k|a_k| < \frac{\varepsilon}{2}. \qquad (9.26)$$

Let $n \ge N$ and $1 - 1/n < x < 1$. Using the right term in (9.26),

$$(1 - x) \sum_{k=0}^{n} k|a_k| < \frac{1}{n} \sum_{k=0}^{n} k|a_k| < \frac{\varepsilon}{2}. \qquad (9.27)$$

Using the left term in (9.26) gives

$$\begin{aligned}
\sum_{k=n+1}^{\infty} |a_k| x^k &< \sum_{k=n+1}^{\infty} \frac{\varepsilon}{2k} x^k \\
&< \frac{\varepsilon}{2n} \frac{x^{n+1}}{1 - x} \qquad (9.28) \\
&< \frac{\varepsilon}{2}.
\end{aligned}$$

Combining (9.26) and (9.26) with (9.25) shows

$$\left| s_n - \sum_{k=0}^{\infty} a_k x^k \right| < \varepsilon.$$

Assumption (b) implies $s_n \to A$. □

9.9 Exercises

Exercise 9.1. If $f_n(x) = nx(1-x)^n$ for $0 \le x \le 1$, then show f_n converges pointwise, but not uniformly on $[0,1]$.

Exercise 9.2. Show $\sin^n x$ converges uniformly on $[0,a]$ for all $a \in (0, \pi/2)$. Does $\sin^n x$ converge uniformly on $[0, \pi/2)$?

Exercise 9.3. Show that $\sum x^n$ converges uniformly on $[-r,r]$ when $0 < r < 1$, but not on $(-1,1)$.

Exercise 9.4. Prove $\sum_{n=0}^{\infty} x^n/n!$ does not converge uniformly on $\mathbb{R}$.

Exercise 9.5. The series

$$\sum_{n=0}^{\infty} \frac{\cos nx}{e^{nx}}$$

is uniformly convergent on any set of the form $[a, \infty)$ with $a > 0$.

Exercise 9.6. A sequence of functions $f_n : S \to \mathbb{R}$ is *uniformly bounded* on S if there is an $M > 0$ such that $\|f_n\|_S \le M$ for all $n \in \mathbb{N}$. Prove that if f_n is uniformly convergent on S and each f_n is bounded on S, then the sequence f_n is uniformly bounded on S.

Exercise 9.7. Let $S \subset \mathbb{R}$ and $c \in \mathbb{R}$. If $f_n : S \to \mathbb{R}$ is a Cauchy sequence, then so is cf_n.

Exercise 9.8. If $S \subset \mathbb{R}$ and $f_n, g_n : S \to \mathbb{R}$ are Cauchy sequences, then so is $f_n + g_n$.

Exercise 9.9. Let $S \subset \mathbb{R}$. If $f_n, g_n : S \to \mathbb{R}$ are uniformly bounded Cauchy sequences, then so is $f_n g_n$.

Exercise 9.10. Prove or give a counterexample: If $f_n : [a,b] \to \mathbb{R}$ is a sequence of monotone functions converging pointwise to a continuous function f, then $f_n \rightrightarrows f$.

Exercise 9.11. Prove there is a sequence of polynomials on $[a,b]$ converging uniformly to a nowhere differentiable function.

Exercise 9.12. Prove Corollary 9.21.

Exercise 9.13. If f is integrable on $[-1,1]$ and continuous at 0, then

$$\lim_{n\to\infty} \int_{-1}^{1} f(t)k_n(t)\,dt = f(0).$$

Exercise 9.14. If $\alpha \in (0,\pi)$, then show that

$$\lim_{n\to\infty} \int_{\alpha}^{\pi} \frac{\sin nx}{nx}\,dx = 0.$$

What about when $\alpha = 0$?

Exercise 9.15. If f is the nowhere differentiable function of Example 9.9, then what is $\int_0^2 f$?

Exercise 9.16. Prove or give a counterexample: If a power series has $(c - R, c + R)$ as its interval of convergence for some $R > 0$ and the power series is absolutely convergent at $x = c - R$, then it is absolutely convergent at $x = c + R$.

Exercise 9.17. Estimate $\int_{-1}^{1} e^{-x^2}\,dx$ to four decimal places.

Exercise 9.18. Suppose a_n, $n \in \omega$, is a bounded sequence of numbers. Show that

$$\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n \tag{9.31}$$

converges to a differentiable function on $\mathbb{R}$ and find its derivative.

Exercise 9.19. Prove $\pi = 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(2n+1)}$. (This is the Madhava-Leibniz series which was used in the fourteenth century to compute π to 11 decimal places.) To how many places must this series be added to get an 11 decimal place approximation of π?

Exercise 9.20. If $\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, then $\frac{d}{dx}\exp(x) = \exp(x)$ for all $x \in \mathbb{R}$.

Exercise 9.21. Is $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ Abel convergent?

Chapter 10

Fourier Series

In the late eighteenth century, it was well-known that complicated functions could sometimes be approximated by a sequence of polynomials. Some of the leading mathematicians at that time, including such luminaries as Daniel Bernoulli, Euler and d'Alembert began studying the possibility of using sequences of trigonometric functions for approximation. In 1807, this idea opened into a huge area of research when Joseph Fourier used series of sines and cosines to solve several outstanding partial differential equations of physics.[1]

In particular, he used series of the form

$$\sum_{n=0}^{\infty} a_n \cos nx + b_n \sin nx$$

to approximate his solutions. Series of this form are called *trigonometric series,* and the ones derived from Fourier's methods are called *Fourier series.* Much of the mathematical research done in the nineteenth and early twentieth century was devoted to understanding the convergence of Fourier series. This chapter presents nothing more than the tip of that huge iceberg.

10.1 Trigonometric Polynomials

Definition 10.1. A function of the form

$$p(x) = \sum_{k=0}^{n} \alpha_k \cos kx + \beta_k \sin kx \tag{10.1}$$

[1]Fourier's methods can be seen in most books on partial differential equations, such as [4]. For example, see solutions of the heat and wave equations using the method of separation of variables.

is called a *trigonometric polynomial*. The largest value of k such that $|\alpha_k| + \beta_k| \neq 0$ is the *degree* of the polynomial. Denote by $\mathcal{T}$ the set of all trigonometric polynomials.

Evidently, all functions in $\mathcal{T}$ are 2π-periodic and $\mathcal{T}$ is closed under addition and multiplication by real numbers. Indeed, it is a real vector space, in the sense of linear algebra and the set $\{\sin nx : n \in \mathbb{N}\} \cup \{\cos nx : n \in \omega\}$ is a basis for $\mathcal{T}$. It is also easy to see $\mathcal{T}$ is closed under differentiation.

The following theorem can be proved using integration by parts or trigonometric identities.

Theorem 10.2. *If $m, n \in \mathbb{Z}$, then*

$$\int_{-\pi}^{\pi} \sin mx \cos nx \, dx = 0, \tag{10.2}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} 0, & m \neq n \\ 0, & m = 0 \text{ or } n = 0 \\ \pi & m = n \neq 0 \end{cases} \tag{10.3}$$

and

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx = \begin{cases} 0, & m \neq n \\ 2\pi & m = n = 0 \\ \pi & m = n \neq 0 \end{cases}. \tag{10.4}$$

If $p(x)$ is as in (10.1), then Theorem 10.2 shows

$$2\pi\alpha_0 = \int_{-\pi}^{\pi} p(x) \, dx,$$

and for $k > 0$,

$$\pi\alpha_k = \int_{-\pi}^{\pi} p(x) \cos kx \, dx, \quad \pi\beta_k = \int_{-\pi}^{\pi} p(x) \sin kx \, dx.$$

Combining these, it follows that if

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} p(x) \cos nx \, dx \text{ and } b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} p(x) \sin nx \, dx$$

for $n \in \omega$, then

$$p(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx. \tag{10.5}$$

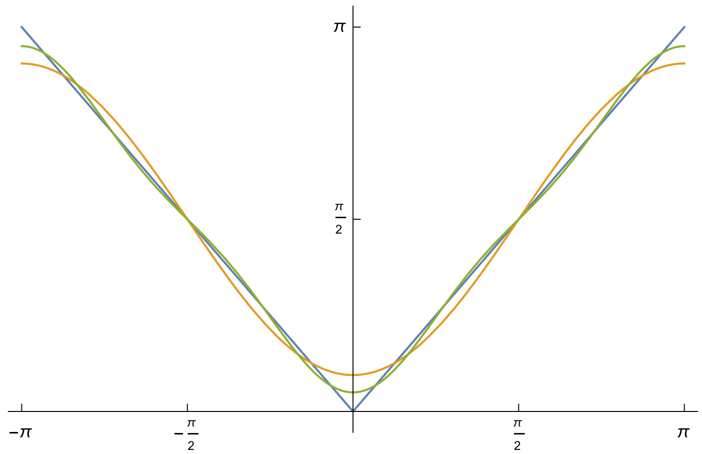

Figure 10.1: This shows $f(x) = |x|$, $s_1(x)$ and $s_3(x)$, where $s_n(x)$ is the n^{th} partial sum of the Fourier series for f.

(Remember that all but a finite number of the a_n and b_n are 0!)

At this point, the logical question is whether this same method can be used to represent a more general 2π-periodic function. For any function f, integrable on $[-\pi, \pi]$, the coefficients can be defined as above; i.e., for $n \in \omega$,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx \text{ and } b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx. \tag{10.6}$$

The numbers a_n and b_n are called the *Fourier coefficients* of f. The problem is whether and in what sense an equation such as (10.5) might be true. This turns out to be a very deep and difficult question with no short answer.[2] Because we don't know whether equality in the sense of (10.5) is true, the usual practice is to write

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx, \tag{10.7}$$

indicating that the series on the right is calculated from the function on the left using (10.6). The series is called the *Fourier series* for f.

[2]Many people, including me, would argue that the study of Fourier series has been the most important area of mathematical research over the past two centuries. Huge mathematical disciplines, including set theory, measure theory and harmonic analysis trace their lineage back to basic questions about Fourier series. Even after centuries of study, research in this area continues unabated.

Example 10.1. Let $f(x) = |x|$. Since f is an even functions and $\sin nx$ is odd,

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \sin nx\, dx = 0$$

for all $n \in \mathbb{N}$. On the other hand,

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx\, dx = \begin{cases} \pi, & n = 0 \\ \dfrac{2(\cos n\pi - 1)}{n^2\pi}, & n \in \mathbb{N} \end{cases}$$

for $n \in \omega$. Therefore,

$$|x| \sim \frac{\pi}{2} - \frac{4\cos x}{\pi} - \frac{4\cos 3x}{9\pi} - \frac{4\cos 5x}{25\pi} - \frac{4\cos 7x}{49\pi} - \frac{4\cos 9x}{81\pi} + \cdots$$

(See Figure 10.1.)

There are at least two fundamental questions arising from (10.7): Does the Fourier series of f converge to f? Can f be recovered from its Fourier series, even if the Fourier series does not converge to f? These are often called the convergence and representation questions, respectively. The next few sections will give some partial answers.

10.2 The Riemann Lebesgue Lemma

We learned early in our study of series that the first and simplest convergence test is to check whether the terms go to zero. For Fourier series, this is always the case.

Theorem 10.3 (Riemann-Lebesgue Lemma). *If f is a function such that $\int_a^b f$ exists, then*

$$\lim_{\alpha\to\infty}\int_a^b f(t)\cos \alpha t\, dt = 0 \text{ and } \lim_{\alpha\to\infty}\int_a^b f(t)\sin \alpha t\, dt = 0.$$

Proof. Since the two limits have similar proofs, only the first will be proved.

Let $\varepsilon > 0$ and P be a generic partition of $[a, b]$ satisfying

$$0 < \int_a^b f - \underline{\mathcal{D}}(f, P) < \frac{\varepsilon}{2}.$$

For $m_i = \text{glb}\{f(x) : x_{i-1} < x < x_i\}$, define a function g on $[a, b]$ by $g(x) = m_i$ when $x_{i-1} \le x < x_i$ and $g(b) = m_n$. Note that $\int_a^b g = \underline{\mathcal{D}}(f, P)$, so

$$0 \le \int_a^b (f - g) < \frac{\varepsilon}{2}. \tag{10.8}$$

Choose

$$\alpha > \frac{4}{\varepsilon}\sum_{i=1}^{n}|m_i|. \tag{10.9}$$

Since $f \geq g$,

$$\begin{aligned}\left|\int_a^b f(t)\cos\alpha t\,dt\right| &= \left|\int_a^b (f(t)-g(t))\cos\alpha t\,dt + \int_a^b g(t)\cos\alpha t\,dt\right|\\ &\leq \left|\int_a^b (f(t)-g(t))\cos\alpha t\,dt\right| + \left|\int_a^b g(t)\cos\alpha t\,dt\right|\\ &\leq \int_a^b (f-g) + \left|\frac{1}{\alpha}\sum_{i=1}^{n} m_i\left(\sin(\alpha x_i) - \sin(\alpha x_{i-1})\right)\right|\\ &\leq \int_a^b (f-g) + \frac{2}{\alpha}\sum_{i=1}^{n}|m_i|\end{aligned}$$

Use (10.8) and (10.9).

$$\begin{aligned}&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\ &= \varepsilon\end{aligned}$$

□

Corollary 10.4. *If f is integrable on $[-\pi,\pi]$ with a_n and b_n the Fourier coefficients of f, then $a_n \to 0$ and $b_n \to 0$.*

10.3 The Dirichlet Kernel

Suppose f is integrable on $[-\pi,\pi]$ and 2π-periodic on $\mathbb{R}$, so the Fourier series of f exists. The partial sums of the Fourier series are written as $s_n(f,x)$, or more simply $s_n(x)$ when there is only one function in sight. To be more precise,

$$s_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n}\left(a_k\cos kx + b_k\sin kx\right).$$

Notice s_n is a trigonometric polynomial of degree at most n.

We begin with the following calculation.

$$
\begin{aligned}
s_n(x) &= \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos kx + b_k \sin kx) \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt + \sum_{k=1}^{n} \frac{1}{\pi} \int_{-\pi}^{\pi} (f(t) \cos kt \cos kx + f(t) \sin kt \sin kx)\; dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(1 + \sum_{k=1}^{n} 2(\cos kt \cos kx + \sin kt \sin kx)\right) dt \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \left(1 + \sum_{k=1}^{n} 2 \cos k(x-t)\right) dt
\end{aligned}
$$

Substitute $s = x - t$ and use the assumption that f is 2π-periodic.

$$
= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-s) \left(1 + 2\sum_{k=1}^{n} \cos ks\right) ds \tag{10.10}
$$

The sequence of trigonometric polynomials from within the integral,

$$
D_n(s) = 1 + 2\sum_{k=1}^{n} \cos ks, \tag{10.11}
$$

is called the *Dirichlet kernel.* Its properties will prove useful for determining the pointwise convergence of Fourier series.

Theorem 10.5. *The Dirichlet kernel has the following properties.*

(a) $D_n(s)$ *is an even* 2π*-periodic function for each* $n \in \mathbb{N}$*.*

(b) $D_n(0) = 2n + 1$ *for each* $n \in \mathbb{N}$*.*

(c) $|D_n(s)| \le 2n + 1$ *for each* $n \in \mathbb{N}$ *and all* s*.*

(d) $\displaystyle \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(s)\, ds = 1$ *for each* $n \in \mathbb{N}$*.*

(e) $\displaystyle D_n(s) = \frac{\sin(n + 1/2)s}{\sin s/2}$ *for each* $n \in \mathbb{N}$ *and* $s/2$ *not an integer multiple of* π*.*

Proof. Properties (a)–(d) follow from the definition of the kernel.

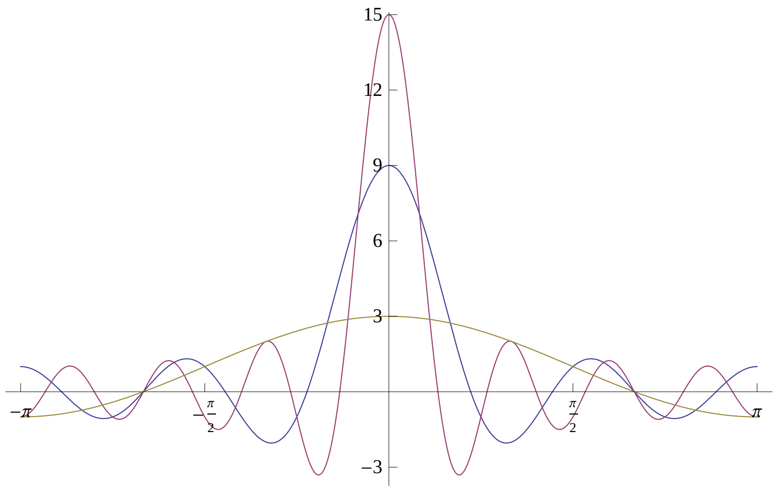

Figure 10.2: The Dirichlet kernel $D_n(s)$ for $n = 1, 4, 7$.

The proof of property (e) uses some trigonometric manipulation. Suppose $n \in \mathbb{N}$ and $s \neq m\pi$ for any $m \in \mathbb{Z}$.

$$D_n(s) = 1 + 2\sum_{k=1}^{n} \cos ks$$

Use the facts that the cosine is even and the sine is odd.

$$\begin{aligned}
&= \sum_{k=-n}^{n} \cos ks + \frac{\cos \frac{s}{2}}{\sin \frac{s}{2}} \sum_{k=-n}^{n} \sin ks \\
&= \frac{1}{\sin \frac{s}{2}} \sum_{k=-n}^{n} \left(\sin \frac{s}{2} \cos ks + \cos \frac{s}{2} \sin ks \right) \\
&= \frac{1}{\sin \frac{s}{2}} \sum_{k=-n}^{n} \sin(k + \frac{1}{2})s
\end{aligned}$$

Since $\sin(-k + 1/2) = -\sin((k-1) + 1/2)$, all but one of the terms in this sum cancel each other.

$$= \frac{\sin\left(n + \frac{1}{2}\right)s}{\sin \frac{s}{2}}$$

□

According to (10.10),

$$s_n(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt.$$

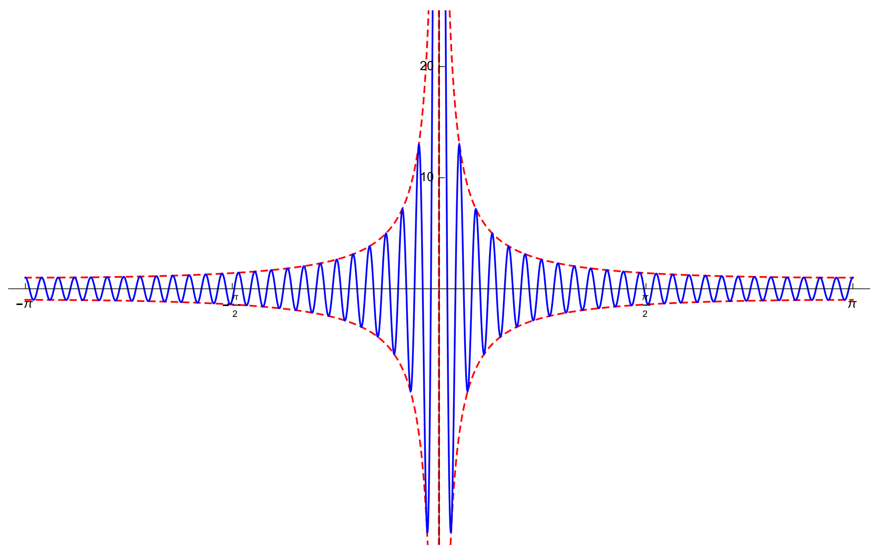

Figure 10.3: This graph shows $D_{50}(t)$ and the envelope $y = \pm 1/\sin(t/2)$. As n gets larger, the $D_n(t)$ fills the envelope more completely.

This is similar to a situation we've seen before within the proof of the Weierstrass approximation theorem, Theorem 9.13. The integral given above is a *convolution integral* similar to that used in the proof of Theorem 9.13, although the Dirichlet kernel isn't a convolution kernel in the sense of Lemma 9.14 because it doesn't satisfy conditions (a) and (c) of that lemma. (See Figure 10.3.)

10.4 Dini's Test for Pointwise Convergence

Theorem 10.6 (Dini's Test)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ be a 2π-periodic function integrable on $[-\pi,\pi]$ with Fourier series given by (10.7). If there is a $\delta > 0$ and $s \in \mathbb{R}$ such that*

$$\int_0^\delta \left| \frac{f(x+t)+f(x-t)-2s}{t} \right| dt < \infty,$$

then

$$\frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kx + b_k \cos kx) = s.$$

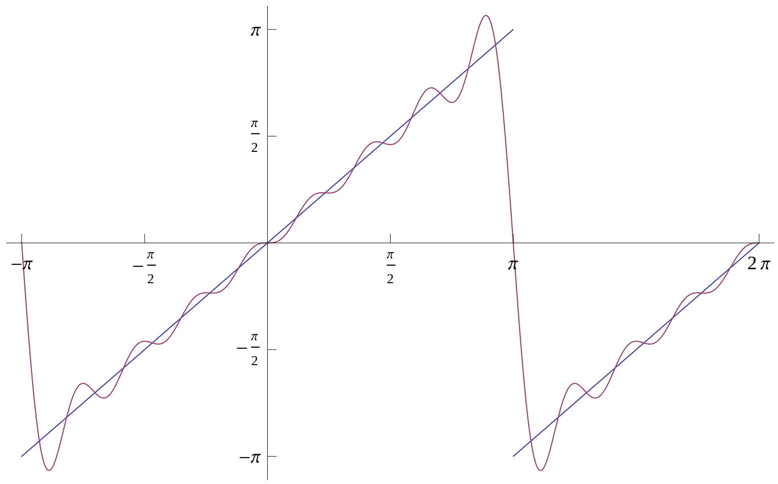

Figure 10.4: This plot shows the function of Example 10.2 and $s_8(x)$ for that function.

Proof. Since D_n is even,

$$\begin{aligned} s_n(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_n(t)\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} f(x-t)D_n(t)\,dt + \frac{1}{2\pi}\int_{0}^{\pi} f(x-t)D_n(t)\,dt \\ &= \frac{1}{2\pi}\int_{0}^{\pi} \left(f(x+t)+f(x-t)\right)D_n(t)\,dt. \end{aligned}$$

By Theorem 10.5(d) and (e),

$$\begin{aligned} s_n(x) - s &= \frac{1}{2\pi}\int_{0}^{\pi} \left(f(x+t)+f(x-t)-2s\right)D_n(t)\,dt \\ &= \frac{1}{2\pi}\int_{0}^{\pi} \frac{f(x+t)+f(x-t)-2s}{t}\cdot\frac{t}{\sin\frac{t}{2}}\cdot\sin(n+\frac{1}{2})t\,dt. \end{aligned}$$

Since $t/\sin\frac{t}{2}$ is bounded on $(0,\pi)$, the Riemann-Lebesgue Lemma (Theorem 10.3) shows $s_n(x) - s \to 0$ as $n \to \infty$. Now use Corollary 8.11 to finish the proof. □

Example 10.2. Suppose $f(x) = x$ for $-\pi < x < \pi$, $f(\pi) = 0$ and f is 2π-periodic on $\mathbb{R}$. This function is often called a *sawtooth wave*. Since f is odd,

$a_n = 0$ for all n. Integration by parts gives $b_n = (-1)^{n+1}2/n$ for $n \in \mathbb{N}$. Therefore,

$$f \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx.$$

(See Figure 10.4.) For $x \in (-\pi, \pi)$, let $0 < \delta < \min\{\pi - x, \pi + x\}$. (This is just the distance from x to closest endpoint of $(-\pi, \pi)$.) Using Dini's test, we see

$$\int_0^\delta \left| \frac{f(x+t) + f(x-t) - 2x}{t} \right| dt = \int_0^\delta \left| \frac{x+t+x-t-2x}{t} \right| dt = 0 < \infty,$$

so

$$\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{n} \sin nx = x \ \text{ for } -\pi < x < \pi. \tag{10.12}$$

In particular, when $x = \pi/2$, (10.12) gives another way to derive (9.24). When $x = \pi$, the series converges to 0, which is the middle of the "jump" for f.

This behavior of converging to the middle of a jump discontinuity is typical. To see this, denote the one-sided limits of f at x by

$$f(x-) = \lim_{t \uparrow x} f(t) \text{ and } f(x+) = \lim_{t \downarrow x} f(t),$$

and suppose f has a jump discontinuity at x with

$$s = \frac{f(x-) + f(x+)}{2}.$$

Guided by Dini's test, consider

$$\begin{aligned}
&\int_0^\delta \left| \frac{f(x+t) + f(x-t) - 2s}{t} \right| dt \\
&\qquad = \int_0^\delta \left| \frac{f(x+t) + f(x-t) - f(x-) - f(x+)}{t} \right| dt \\
&\qquad \le \int_0^\delta \left| \frac{f(x+t) - f(x+)}{t} \right| dt + \int_0^\delta \left| \frac{f(x-t) - f(x-)}{t} \right| dt
\end{aligned}$$

If both of the integrals on the right are finite, then the integral on the left is also finite. This amounts to a proof of the following corollary.

Corollary 10.7. *Suppose $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and integrable on $[-\pi, \pi]$. If both one-sided limits exist at x and there is a $\delta > 0$ such that both*

$$\int_0^\delta \left| \frac{f(x+t) - f(x+)}{t} \right| dt < \infty \text{ and } \int_0^\delta \left| \frac{f(x-t) - f(x-)}{t} \right| dt < \infty,$$

then the Fourier series of f converges to

$$\frac{f(x-) + f(x+)}{2}.$$

The Dini test given above provides a powerful condition sufficient to ensure the pointwise convergence of a Fourier series. One would be tempted to think a consequence is that the Fourier series of a continuous function must converge to that function at points of continuity. Unfortunately, this is not true. In Section 10.6, a continuous function whose Fourier series diverges is constructed. In fact, there are integrable functions with everywhere divergent Fourier series [16].

In addition to Dini's test, there is a plethora of ever more abstruse conditions that can be proved in a similar fashion to show pointwise convergence.

10.5 Gibbs Phenomenon

For $x \in [-\pi, \pi)$ define

$$f(x) = \begin{cases} \frac{|x|}{x}, & 0 < |x| < \pi \\ 0, & x = -\pi, 0 \end{cases} \tag{10.13}$$

and extend f 2π-periodically to all of $\mathbb{R}$. This function is often called a *square wave.* A straightforward calculation gives

$$f \sim \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)x}{2k-1}.$$

Corollary 10.7 shows $s_n(x) \to f(x)$ everywhere. This convergence cannot be uniform because all the partial sums are continuous and f is discontinuous at every integer multiple of π. A plot of $s_{19}(x)$ is shown in Figure 10.5. Notice the higher peaks in the oscillation of $s_n(x)$ just before and after the jump discontinuities of f. This behavior is not unique to f, as it can also be seen in Figure 10.4. If a function has different one-sided limits at a point, the partial

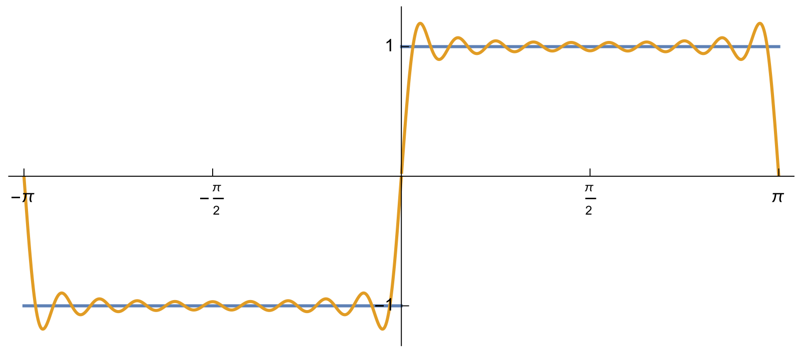

Figure 10.5: This is a plot of $s_{19}(f,x)$, where f is defined by (10.13).

sums of its Fourier series will always have such higher peaks near that point. This behavior is called *Gibbs phenomenon*.[3]

Instead of doing a general analysis of Gibbs phenomenon, we'll only analyze the simple case shown in the square wave f. It's basically a calculus exercise.

To locate the peaks in the graph, differentiate the partial sums.

$$s'_{2n-1}(x) = \frac{d}{dx}\frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin(2k-1)x}{2k-1} = \frac{4}{\pi}\sum_{k=1}^{n}\cos(2k-1)x$$

It is left as Exercise 13 to show this has a closed form.

$$s'_{2n-1}(x) = \frac{2}{\pi}\frac{\sin 2nx}{\sin x}$$

Looking at the numerator, we see $s'_{2n-1}(x)$ has critical numbers at $x = k\pi/2n$ for $k \in \mathbb{Z}$. In the interval $(0,\pi)$, $s_{2n-1}(k\pi/2n)$ is a relative maximum for odd k and a relative minimum for even k. (See Figure 10.6.) The value $s_{2n-1}(\pi/2n)$ is the height of the left-most peak. What is the behavior of these maxima?

[3] It is named after the American mathematical physicist, J. W. Gibbs, who pointed it out in 1899. He was not the first to notice the phenomenon, as the British mathematician Henry Wilbraham had published a little-noticed paper on it in 1848. Gibbs' interest in the phenomenon was sparked by investigations of the American experimental physicist A. A. Michelson who wrote a letter to *Nature* disputing the possibility that a Fourier series could converge to a discontinuous function. The ensuing imbroglio is recounted in a marvelous book by Paul J. Nahin [19].

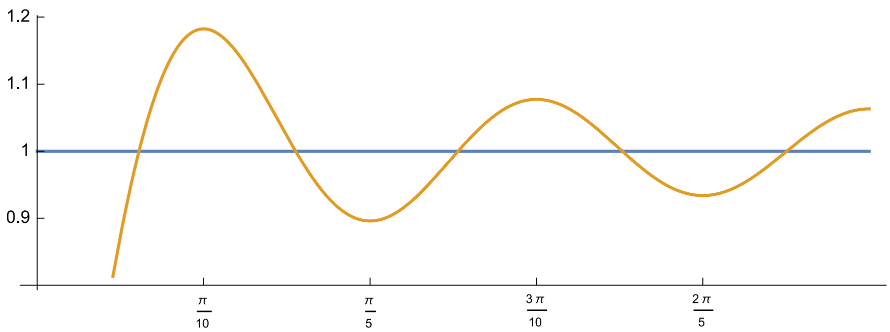

Figure 10.6: This is a plot of $s_9(f, x)$, where f is defined by (10.13).

From Figure 10.7 it appears they have an asymptotic limit near 1.18. To prove this, consider the following calculation.

$$\begin{aligned} s_{2n-1}\left(\frac{\pi}{2n}\right) &= \frac{4}{\pi}\sum_{k=1}^{n}\frac{\sin\left((2k-1)\frac{\pi}{2n}\right)}{2k-1} \\ &= \frac{2}{\pi}\sum_{k=1}^{n}\frac{\sin\left(\frac{(2k-1)\pi}{2n}\right)}{\frac{(2k-1)\pi}{2n}}\frac{\pi}{n} \end{aligned}$$

The last sum is a midpoint Riemann sum for the function $\sin x/x$ on the interval $[0, \pi]$ using a regular partition of n subintervals. Example 9.16 shows

$$\frac{2}{\pi}\int_0^{\pi}\frac{\sin x}{x}\,dx \approx 1.17898.$$

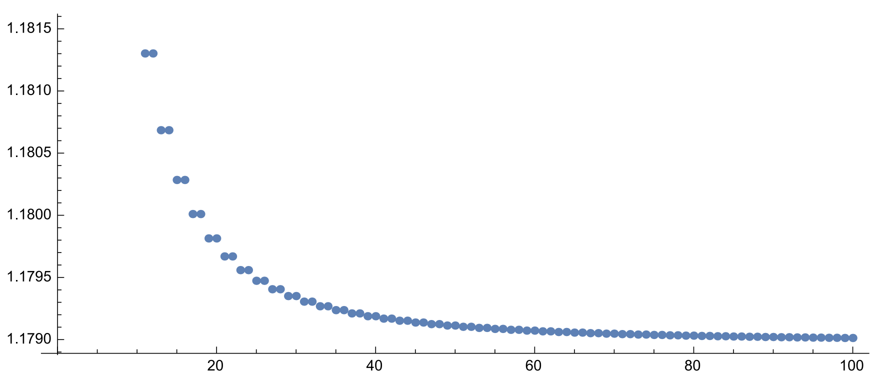

Figure 10.7: This is a plot of $s_n(f, \pi/2n)$ for $n = 1, 2, \cdots, 100$. The dots come in pairs because $s_{2n-1}(f, \pi/2n) = s_{2n}(f, \pi/2n)$.

Since $f(0+) - f(0-) = 2$, this is an overshoot of a bit less than 9%. There is a similar undershoot on the other side. It turns out this is typical behavior at points of discontinuity [23].

10.6 A Divergent Fourier Series

Nineteenth century mathematicians, including Fourier, Cauchy, Euler, Weierstrass, Lagrange and Riemann, had computed the Fourier series for many functions. They believed from these examples that the Fourier series of a continuous function must converge to that function. Fourier went far beyond this claim in his important 1822 book *Théorie Analytique de la Chaleur,* by asserting that any function could be represented by linear combinations of functions of the form $\sin nt$ and $\cos nt$. Cauchy even published a flawed proof of this "fact." It took almost a century and a half to develop a clear understanding of what's really going on.

In 1873, Paul du Bois-Reymond, partially settled the question by giving the construction of a continuous function whose Fourier series diverges at a point [2]. It was finally shown in 1966 by Lennart Carleson [8] that the Fourier series of a continuous function converges to that function everywhere with the exception of a set of measure zero. At nearly the same time Kahane and Katznelson [15] showed that given any set of measure zero there is a continuous function whose Fourier series diverges on that set.

The problems around the convergence of Fourier series motivated a huge amount of research that is still going on today. In this section, we look at the tip of that iceberg by presenting a continuous function $F(t)$ whose Fourier series fails to converge when $t = 0$.

10.6.1 The Conjugate Dirichlet Kernel

Lemma 10.8. *If $m, n \in \omega$ and $0 < |t| < \pi$ for $k \in \mathbb{Z}$, then*

$$\sum_{k=m}^{n} \sin kt = \frac{\cos(m - \frac{1}{2})t - \cos(m + \frac{1}{2})t}{2 \sin \frac{t}{2}}.$$

Proof.

$$\begin{aligned}\sum_{k=m}^{n} \sin kt &= \frac{1}{\sin \frac{t}{2}} \sum_{k=m}^{n} \sin \frac{t}{2} \sin kt \\ &= \frac{1}{\sin \frac{t}{2}} \sum_{k=m}^{n} \left(\cos \left(k - \frac{1}{2} \right) t - \cos \left(k + \frac{1}{2} \right) t \right) \\ &= \frac{\cos \left(m - \frac{1}{2} \right) t - \cos \left(n + \frac{1}{2} \right) t}{2 \sin \frac{t}{2}}\end{aligned}$$

□

Definition 10.9. The *conjugate Dirichlet kernel*[4] is

$$\tilde{D}_n(t) = \sum_{k=1}^{n} \sin kt, \quad n \in \mathbb{N}. \tag{10.14}$$

We'll not have much use for the conjugate Dirichlet kernel, except as a convenient way to refer to sums of the form (10.14).

Lemma 10.8 immediately gives the following bound.

Corollary 10.10. *If* $0 < |t| < \pi$*, then*

$$\left| \tilde{D}_n(t) \right| \leq \frac{1}{\left| \sin \frac{t}{2} \right|}.$$

10.6.2 A Sawtooth Wave

If the function $f(x) = (\pi - x)/2$ on $[0, 2\pi)$ is extended 2π-periodically to $\mathbb{R}$, then the graph of the resulting function is a sawtooth wave. (See Example 10.2.) It has a particularly nice Fourier series:

$$\frac{\pi - x}{2} \sim \sum_{k=1}^{\infty} \frac{\sin kx}{k}.$$

According to Corollary 10.7

$$\sum_{k=1}^{\infty} \frac{\sin kx}{k} = \begin{cases} 0, & x = 2n\pi,\ n \in \mathbb{Z} \\ f(x), & \text{otherwise} \end{cases}.$$

We're interested in various partial sums of this series.

[4] In this case, the word "conjugate" does not refer to the complex conjugate, but to the harmonic conjugate. They are related by $D_n(t) + i\tilde{D}_n(t) = 1 + 2\sum_{k=1}^{n} e^{kit}$.

Lemma 10.11. *If $m, n \in \omega$ with $m \le n$ and $0 < |t| < 2\pi$, then*

$$\left|\sum_{k=m}^{n} \frac{\sin kt}{k}\right| \le \frac{1}{m\left|\sin \frac{t}{2}\right|}.$$

Proof.

$$\left|\sum_{k=m}^{n} \frac{\sin kt}{k}\right| = \left|\sum_{k=m}^{n} \left(\tilde{D}_k(t) - \tilde{D}_{k-1}(t)\right)\frac{1}{k}\right|$$

Use summation by parts.

$$\begin{aligned}
&= \left|\sum_{k=m}^{n} \tilde{D}_k(t)\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{\tilde{D}_n(t)}{n+1} - \frac{\tilde{D}_{n-1}(t)}{m}\right| \\
&\le \left|\sum_{k=m}^{n} \tilde{D}_k(t)\left(\frac{1}{k} - \frac{1}{k+1}\right)\right| + \left|\frac{\tilde{D}_n(t)}{n+1}\right| + \left|\frac{\tilde{D}_{n-1}(t)}{m}\right|
\end{aligned}$$

Apply Corollary 10.10.

$$\begin{aligned}
&\le \frac{1}{2\sin\frac{t}{2}}\left(\sum_{k=m}^{n}\left(\frac{1}{k} - \frac{1}{k+1}\right) + \frac{1}{n+1} + \frac{1}{m}\right) \\
&= \frac{1}{2\sin\frac{t}{2}}\left(\frac{1}{m} - \frac{1}{n+1} + \frac{1}{n+1} + \frac{1}{m}\right) \\
&= \frac{1}{m\sin\frac{t}{2}}.
\end{aligned}$$

□

Proposition 10.12. *If $n \in \mathbb{N}$ and $0 < |t| < \pi$, then*

$$\left|\sum_{k=1}^{n} \frac{\sin kt}{k}\right| \le 1 + \pi.$$

Proof.

$$\begin{aligned}
\left|\sum_{k=1}^{n}\frac{\sin kt}{k}\right| &= \left|\sum_{1\le k\le \frac{1}{t}}\frac{\sin kt}{k} + \sum_{\frac{1}{t}<k\le n}\frac{\sin kt}{k}\right| \\
&\le \left|\sum_{1\le k\le \frac{1}{t}}\frac{\sin kt}{k}\right| + \left|\sum_{\frac{1}{t}<k\le n}\frac{\sin kt}{k}\right| \\
&\le \sum_{1\le k\le \frac{1}{t}}\frac{kt}{k} + \frac{1}{\frac{1}{t}\sin\frac{t}{2}} \\
&\le \sum_{1\le k\le \frac{1}{t}} t + \frac{1}{\frac{1}{t}\frac{2}{\pi}\frac{t}{2}} \\
&\le 1+\pi
\end{aligned}$$

□

10.6.3 A Continuous Function with a Divergent Fourier Series

The goal in this subsection is to construct a continuous function F such that

$$\limsup s_n(F,0) = \infty.$$

This implies $s_n(F,0)$ does not converge.

Example 10.3. The first step in the construction is to define a sequence of trigonometric polynomials that form the building blocks for F.

$$\begin{aligned}
f_n(t) = \frac{1}{n} + \frac{\cos t}{n-1} + \frac{\cos 2t}{n-2} + \cdots + \frac{\cos(n-1)t}{1} \\
- \frac{\cos(n+1)t}{1} - \frac{\cos(n+2)t}{2} - \cdots - \frac{\cos 2nt}{n}
\end{aligned}$$

Three facts about partial sums of f_n will be important in what follows.

Since f_n is a trigonometric polynomial of degree $2n$, direct substitution shows

$$m \ge 2n \implies s_m(f_n,0) = f_n(0) = 0. \tag{10.15}$$

Noting which terms cancel when $t=0$, we see

$$1 \le m < 2n \implies s_m(f_n,0) > 0. \tag{10.16}$$

Finally,

$$0 \le m < n \implies s_m(f_n, 0) = \sum_{k=0}^{m} \frac{1}{n-k}. \tag{10.17}$$

Rearrange the sum in the definition of f_n and use the cosine sum identity to see

$$\begin{aligned} f_n(t) &= \sum_{k=1}^{n} \frac{\cos(n-k)t - \cos(n+k)t}{k} \\ &= 2\sin nt \sum_{k=1}^{n} \frac{\sin kt}{k}. \end{aligned}$$

This closed form for f_n combined with Proposition 10.12 implies the sequence of functions f_n is uniformly bounded:

$$|f_n(t)| = \left| 2\sin nt \sum_{k=1}^{n} \frac{\sin kt}{k} \right| \le 2 + 2\pi. \tag{10.18}$$

At last, the main function can be defined.

$$F(t) = \sum_{n=1}^{\infty} \frac{f_{2^{n^3}}(t)}{n^2}$$

The Weierstrass M-Test along with (10.18) implies F is uniformly convergent and therefore continuous on $\mathbb{R}$. Consider

$$\begin{aligned} s_{2^{m^3}}(F, 0) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} F(t) D_{2^{m^3}}(t)\, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} \frac{f_{2^{n^3}}(t)}{n^2} D_{2^{m^3}}(t)\, dt \end{aligned}$$

The uniform convergence allows the sum and integration to be reordered.

$$\begin{aligned} &= \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_{2^{n^3}}(t) D_{2^{m^3}}(t)\, dt \\ &= \sum_{n=1}^{\infty} \frac{1}{n^2} s_{2^{m^3}}\left(f_{2^{n^3}}, 0\right) \end{aligned}$$

Use (10.15), (10.16), (10.17) and the fact that $\sum_{k=1}^{n} 1/k > \ln n$ (cf. Exercise 11).

$$\begin{aligned}
&= \sum_{n=m}^{\infty} \frac{1}{n^2} s_{2^{m^3}}\left(f_{2^{n^3}}, 0\right) \\
&> \frac{1}{m^2} s_{2^{m^3}}\left(f_{2^{m^3}}, 0\right) \\
&= \frac{1}{m^2} \sum_{k=1}^{2^{m^3}} \frac{1}{k} \\
&> \frac{1}{m^2} \ln 2^{m^3} \\
&= m \ln 2
\end{aligned}$$

This implies, $\limsup s_n(F, 0) \geq \lim_{m\to\infty} m \ln 2 = \infty$, so $s_n(F, 0)$ does not converge.

10.7 The Fejér Kernel

The basic question here is how to recover a function from its Fourier coefficients. In the previous several sections, we've seen that pointwise convergence of the partial sums won't always work—even for continuous functions. A different method is required. Instead of looking at the sequence of partial sums, consider a rolling average instead:

$$\sigma_n(f, x) = \frac{1}{n+1} \sum_{k=0}^{n} s_n(f, x).$$

The trigonometric polynomials $\sigma_n(f, x)$ are called the Cesàro means of the partial sums.[5] If $\lim_{n\to\infty} \sigma_n(f, x)$ exists, then the Fourier series for f is said to be $(C, 1)$ *summable* at x. The idea is that this averaging will "smooth out" the partial sums, making them more nicely behaved. It is not hard to show that if $s_n(f, x)$ converges at some x, then $\sigma_n(f, x)$ will converge to the same thing. But there are sequences for which $\sigma_n(f, x)$ converges and $s_n(f, x)$ does not (cf. Exercises 27 and 28).

As with $s_n(x)$, we'll simply write $\sigma_n(x)$ instead of $\sigma_n(f, x)$, when it is clear which function is being considered.

We start with a calculation.

[5]Ernesto Cesàro, 1859-1906, was an Italian mathematician. He introduced the (C, n) summation methods in 1890.

$$
\begin{aligned}
\sigma_n(x) &= \frac{1}{n+1}\sum_{k=0}^{n} s_k(x) \\
&= \frac{1}{n+1}\sum_{k=0}^{n}\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)D_k(t)\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1}{n+1}\sum_{k=0}^{n} D_k(t)\,dt \qquad (*) \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1}{n+1}\sum_{k=0}^{n}\frac{\sin(k+1/2)t}{\sin t/2}\,dt \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1}{(n+1)\sin^2 t/2}\sum_{k=0}^{n}\sin t/2\,\sin(k+1/2)t\,dt
\end{aligned}
$$

Use the identity $2\sin A\sin B = \cos(A-B)-\cos(A+B)$.

$$
= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1/2}{(n+1)\sin^2 t/2}\sum_{k=0}^{n}\left(\cos kt-\cos(k+1)t\right)\,dt
$$

The sum telescopes.

$$
= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1/2}{(n+1)\sin^2 t/2}\left(1-\cos(n+1)t\right)\,dt
$$

Use the identity $2\sin^2 A = 1-\cos 2A$.

$$
= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-t)\frac{1}{(n+1)}\left(\frac{\sin\frac{n+1}{2}t}{\sin\frac{t}{2}}\right)^2\,dt \qquad (**)
$$

The *Fejér kernel* is the sequence of functions highlighted above;[6] i.e.,

$$
K_n(t) = \frac{1}{(n+1)}\left(\frac{\sin\frac{n+1}{2}t}{\sin\frac{t}{2}}\right)^2,\ n\in\mathbb{N}. \qquad (10.19)
$$

[6]Lipót Fejér, 1880-1959, was a Hungarian mathematician. He introduced the Fejér kernel in 1904.

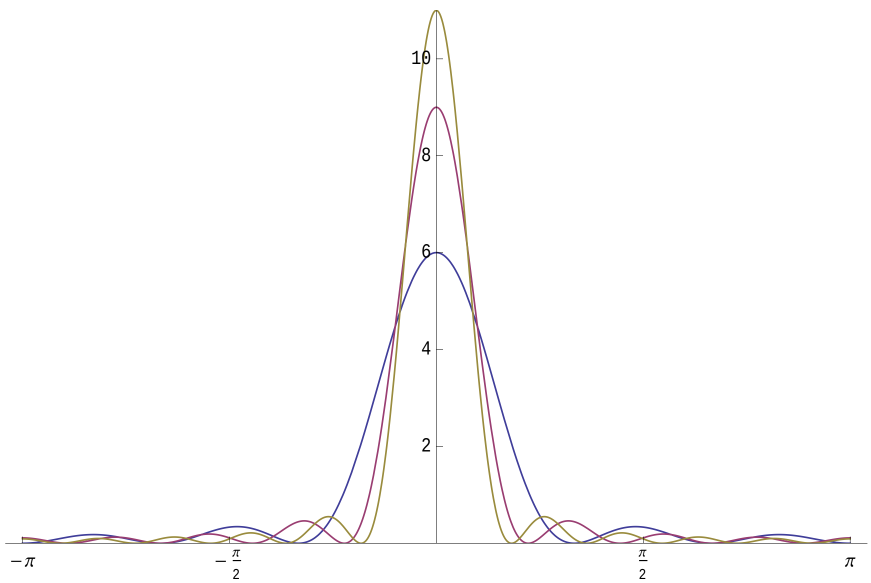

Figure 10.8: A plot of $K_5(t)$, $K_8(t)$ and $K_{10}(t)$.

Comparing the lines labeled (*) and (**) in the previous calculation, we see another form for the Fejér kernel is

$$K_n(t) = \frac{1}{n+1} \sum_{k=0}^{n} D_k(t). \tag{10.20}$$

Once again, we're confronted with a convolution integral containing a kernel:

$$\sigma_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_n(t)\, dt.$$

Theorem 10.13. *The Fejér kernel has the following properties.*[7]

(a) $K_n(t)$ *is an even* 2π*-periodic function for each* $n \in \mathbb{N}$.

(b) $K_n(0) = n + 1$ *for each* $n \in \omega$.

(c) $K_n(t) \geq 0$ *for each* $n \in \mathbb{N}$.

(d) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(t)\, dt = 1$ *for each* $n \in \omega$.

[7] Compare this theorem with Lemma 9.14.

(e) *If* $0 < \delta < \pi$, *then* $K_n \rightrightarrows 0$ *on* $[-\pi, \delta] \cup [\delta, \pi]$.

(f) *If* $0 < \delta < \pi$, *then* $\int_{-\pi}^{\delta} K_n(t)\,dt \to 0$ *and* $\int_{\delta}^{\pi} K_n(t)\,dt \to 0$.

Proof. Theorem 10.5 and (10.20) imply (a), (b) and (d). Equation (10.19) implies (c).

Let δ be as in (e). In light of (a), it suffices to prove (e) for the interval $[\delta, \pi]$. Noting that

$$0 < \sin(\delta/2) \le \sin(t/2) \le 1$$

on $[\delta, \pi]$, it follows that for $\delta \le t \le \pi$,

$$\begin{aligned} K_n(t) &= \frac{1}{(n+1)} \left(\frac{\sin \frac{n+1}{2} t}{\sin \frac{t}{2}} \right)^2 \\ &\le \frac{1}{(n+1)} \left(\frac{1}{\sin \frac{t}{2}} \right)^2 \\ &\le \frac{1}{(n+1)} \frac{1}{\sin^2 \frac{\delta}{2}} \to 0 \end{aligned}$$

It follows that $K_n \rightrightarrows 0$ on $[\delta, \pi]$ and (e) has been proved.

Theorem 9.16 and (e) imply (f). □

Theorem 10.14 (Fejér). *If* $f : \mathbb{R} \to \mathbb{R}$ *is* 2π*-periodic, integrable on* $[-\pi, \pi]$ *and continuous at* x, *then* $\sigma_n(x) \to f(x)$.

Proof. Since f is 2π-periodic and $\int_{-\pi}^{\pi} f(t)\,dt$ exists, so does $\int_{-\pi}^{\pi} \int (f(x-t) - f(x))\,dt$. Theorem 8.3 gives an $M > 0$ so $|f(x-t) - f(x)| < M$ for all t.

Let $\varepsilon > 0$ and choose $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon/3$ whenever $|x - y| < \delta$. By Theorem 10.13(f), there is an $N \in \mathbb{N}$ so that whenever $n \ge N$,

$$\frac{1}{2\pi} \int_{-\pi}^{\delta} K_n(t)\,dt < \frac{\varepsilon}{3M} \text{ and } \frac{1}{2\pi} \int_{\delta}^{\pi} K_n(t)\,dt < \frac{\varepsilon}{3M}.$$

We start calculating.

$$\begin{aligned}
|\sigma_n(x) - f(x)| &= \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) K_n(t)\,dt - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) K_n(t)\,dt \right| \\
&= \frac{1}{2\pi} \left| \int_{-\pi}^{\pi} (f(x-t) - f(x)) K_n(t)\,dt \right| \\
&= \frac{1}{2\pi} \left| \int_{-\pi}^{-\delta} (f(x-t) - f(x)) K_n(t)\,dt + \int_{-\delta}^{\delta} (f(x-t) - f(x)) K_n(t)\,dt \right. \\
&\qquad \left. + \int_{\delta}^{\pi} (f(x-t) - f(x)) K_n(t)\,dt \right| \\
&\le \left| \frac{1}{2\pi} \int_{-\pi}^{-\delta} (f(x-t) - f(x)) K_n(t)\,dt \right| + \left| \frac{1}{2\pi} \int_{-\delta}^{\delta} (f(x-t) - f(x)) K_n(t)\,dt \right| \\
&\qquad + \left| \frac{1}{2\pi} \int_{\delta}^{\pi} (f(x-t) - f(x)) K_n(t)\,dt \right| \\
&< \frac{M}{2\pi} \int_{-\pi}^{-\delta} K_n(t)\,dt + \frac{1}{2\pi} \int_{-\delta}^{\delta} |f(x-t) - f(x)| K_n(t)\,dt + \frac{M}{2\pi} \int_{\delta}^{\pi} K_n(t)\,dt \\
&< M \frac{\varepsilon}{3M} + \frac{\varepsilon}{3} \frac{1}{2\pi} \int_{-\delta}^{\delta} K_n(t)\,dt + M \frac{\varepsilon}{3M} < \varepsilon
\end{aligned}$$

This shows $\sigma_n(x) \to f(x)$. □

Theorem 10.14 gives a partial solution to the representation problem.

Corollary 10.15. *Suppose f and g are continuous and 2π-periodic on $\mathbb{R}$. If f and g have the same Fourier coefficients, then they are equal.*

Proof. By assumption, $\sigma_n(f, t) = \sigma_n(g, t)$ for all n and Theorem 10.14 implies

$$0 = \sigma_n(f, t) - \sigma_n(g, t) \to f - g.$$

□

In the case of continuous functions, the convergence is uniform, rather than pointwise.

Theorem 10.16 (Fejér). *If $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and continuous, then $\sigma_n(x) \rightrightarrows f(x)$.*

Proof. By Exercise 33, f is uniformly continuous. This can be used to show the calculation within the proof of Theorem 10.14 does not depend on x. The details are left as Exercise 15. □

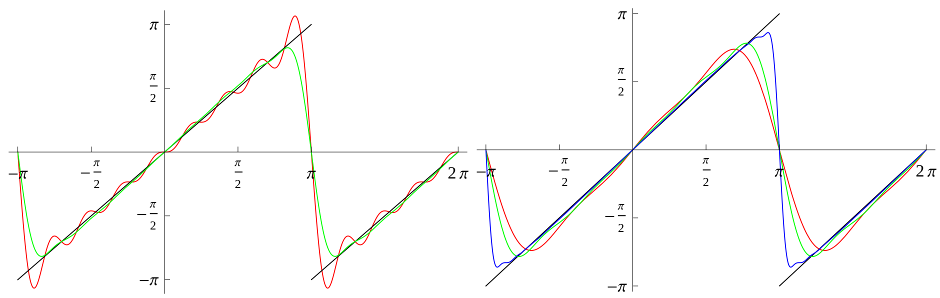

Figure 10.9: These plots illustrate the functions of Example 10.4. On the left are shown $f(x)$, $s_8(x)$ and $\sigma_8(x)$. On the right are shown $f(x)$, $\sigma_3(x)$, $\sigma_5(x)$ and $\sigma_{20}(x)$. Compare this with Figure 10.4.

A perspicacious reader will have noticed the similarity between Theorem 10.16 and the Weierstrass Approximation Theorem 9.13. In fact, the Weierstrass Approximation Theorem can be proved from Theorem 10.16 using power series and Theorem 9.24. (See Exercise 16.)

Example 10.4. As in Example 10.2, let $f(x) = x$ for $-\pi < x \le \pi$ and extend f to be periodic on $\mathbb{R}$ with period 2π. Figure 10.9 shows the difference between the Fejér and classical methods of summation. Notice that the Fejér sums remain much more smoothly affixed to the function and do not show Gibbs phenomenon.

10.8 Exercises

Exercise 10.1. Prove Theorem 10.2.

Exercise 10.2. Suppose $f : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and integrable on $[-\pi, \pi]$. If $b - a = 2\pi$, then $\int_a^b f = \int_{-\pi}^{\pi} f$.

Exercise 10.3. Let $f : [-\pi, \pi) \to \mathbb{R}$ be a function with Fourier coefficients a_n and b_n. If f is odd, then $a_n = 0$ for all $n \in \omega$. If f is even, then $b_n = 0$ for all $n \in \mathbb{N}$.

Exercise 10.4. If $f(x) = \operatorname{sgn}(x)$ on $[-\pi, \pi)$, then find the Fourier series for f.

Exercise 10.5. Is $\displaystyle\sum_{n=1}^{\infty} \sin nx$ the Fourier series of some function?

Exercise 10.6. Let $f(x) = x^2$ when $-\pi \leq x < \pi$ and extend f to be 2π-periodic on all of $\mathbb{R}$. Use Dini's Test to show $s_n(f,x) \to f(x)$ everywhere.

Exercise 10.7. Suppose f is a 2π-periodic function such that

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Prove

$$\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \leq \int_{-\pi}^{\pi} f^2.$$

Exercise 10.8. Let $f(x) = \pi - |x|$ on $[-\pi, \pi]$ and extend f periodically to all of $\mathbb{R}$. Show the Fourier series of f is uniformly convergent on $\mathbb{R}$.

Exercise 10.9. Use Exercise 6 to prove $\dfrac{\pi^2}{6} = \sum_{n=1}^{\infty} \dfrac{1}{n^2}$.

Exercise 10.10. Use the Fourier series for $|x|$ to prove

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Exercise 10.11. If

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx \rightrightarrows f \tag{10.22}$$

on $[-\pi, \pi)$, then the series on the left side of (10.22) is the Fourier series for f.

Exercise 10.12. If $f : \mathbb{R} \to \mathbb{R}$ is periodic with period p and continuous on $[0, p]$, then f is uniformly continuous.

Exercise 10.13. Prove

$$\sum_{k=1}^{n} \cos(2k-1)t = \frac{\sin 2nt}{2 \sin t}$$

for $t \neq k\pi$, $k \in \mathbb{Z}$ and $n \in \mathbb{N}$. (Hint: $2\sum_{k=1}^{n} \cos(2k-1)t = D_{2n}(t) - D_n(2t)$.)

Exercise 10.14. The function $g(t) = t/\sin(t/2)$ is undefined whenever $t = 2n\pi$ for some $n \in \mathbb{Z}$. Show that it can be redefined on the set $\{2n\pi : n \in \mathbb{Z}\}$ to be periodic and uniformly continuous on $\mathbb{R}$.

Exercise 10.15. Prove Theorem 10.16.

Exercise 10.16. Prove the Weierstrass approximation theorem using Fourier series and Taylor's theorem.

Exercise 10.17. If $f(x) = x$ for $-\pi \le x < \pi$, can the Fourier series of f be integrated term-by-term on $(-\pi, \pi)$ to obtain the correct answer?

Exercise 10.18. Suppose f is a 2π-periodic function such that

$$f \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx.$$

Prove

$$\left(\frac{a_0}{2}\right)^2 + \sum_{n=1}^{\infty} a_n^2 + b_n^2 \le \int_{-\pi}^{\pi} f^2.$$

This is called Bessel's inequality.

Exercise 10.19. Suppose f is 2π-periodic, integrable and decreasing on $[0, 2\pi)$. Show that $f \sim \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$, where $b_n \ge 0$ for all n.

The following exercises explore the relationship between Fourier series and differentiation.

Exercise 10.20. Let $f : \mathbb{R} \to \mathbb{R}$ be 2π-periodic and integrable on $[-\pi, \pi]$. If f is differentiable at some point, then the Fourier series for f converges to f at that point.

Exercise 10.21. If $f(x) = x$ for $-\pi \le x < \pi$, can the Fourier series of f be differentiated term-by-term on $(-\pi, \pi)$ to obtain the correct answer?

Exercise 10.22. Let f be a 2π-periodic function such that f' is continuous. If a_n and b_n are the Fourier coefficients of f, then

$$\sum_{n=1}^{\infty} |a_n| < \infty \text{ and } \sum_{n=1}^{\infty} |b_n| < \infty.$$

Moreover, $na_n \to 0$ and $nb_n \to 0$.

Exercise 10.23. If f satisfies the conditions of Exercise 22, then the Fourier series of f converges absolutely and to f uniformly.

Exercise 10.24. Let f be 2π-periodic with a continuous second derivative. Then

$$f'(x) = \sum_{n=1}^{\infty} nb_n \cos nx - na_n \sin nx$$

and this is the Fourier series of f'.

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