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Travelling wave solutions in delayed cooperative systems

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Abstract
We establish the existence of travelling wave solutions for delayed cooperative recursions that are allowed to have more than two equilibria. We define an important extended real number that is used to determine the speeds of travelling wave solutions. The results can be applied to a large class of delayed cooperative reaction–diffusion models. We show that for a delayed Lotka–Volterra reaction–diffusion competition model, there exists a finite positive number $c_+^*$ that can be characterized as the slowest speed of travelling wave solutions connecting two mono-culture equilibria or connecting a mono-culture with the coexistence equilibrium.

Mathematics Subject Classification: 92D40, 92D25, 35K55, 35K57

1. Introduction

This work is motivated by the study of the existence of travelling wave solutions in the Lotka–Volterra-type two-species competition model with delay

$$\begin{align*}
\frac{\partial u}{\partial t} &= d_u \frac{\partial^2 u}{\partial x^2} + \alpha_1 \int_{\mathbb{R}} G_u(y) u(t - \tau_1, x - y) \, dy - \eta_1 u^2 - p_1 uv, \\
\frac{\partial v}{\partial t} &= d_v \frac{\partial^2 v}{\partial x^2} + \alpha_2 \int_{\mathbb{R}} G_v(y) v(t - \tau_2, x - y) \, dy - \eta_2 v^2 - p_2 uv.
\end{align*}$$

(1.1)

Here $u(t, x)$ and $v(t, x)$ represent densities of adult members of two species $u$ and $v$ at time $t$ and point $x$, respectively. $d_u > 0$ ($d_v > 0$) is the diffusion coefficient of the adult population $u$ ($v$). The delay $\tau_1$ ($\tau_2$) describes the time taken from birth to maturity of the population $u$ ($v$). $\alpha_1$ ($\alpha_2$) is made up of two factors, the per capita birth rate and the survival rate of immatures.
for the population $u$ ($v$) during the immature stage. The two probability kernels $G_u$ and $G_v$ are given by

$$G_u(y) = \frac{e^{-y^2/4d_i(u)\tau_1}}{\sqrt{4\pi d_i(u)\tau_1}}, \quad G_v(y) = \frac{e^{-y^2/4d_i(v)\tau_2}}{\sqrt{4\pi d_i(v)\tau_2}},$$  \hspace{1cm} (1.2)

where $d_i(u) > 0$ ($d_i(v) > 0$) is the diffusion coefficient of the immature population of $u$ ($v$).

The terms $\alpha_1 \int_{\mathbb{R}} G_u(y)u(t - \tau_1, x - y) dy$ and $\alpha_2 \int_{\mathbb{R}} G_v(y)u(t - \tau_2, x - y) dy$ total up the individuals of the populations $u$ and $v$ born at time $t - \tau_1$ and $t - \tau_2$ in all parts of the domain that are still alive at time $t$ and have just reached maturity and arrived at $x$, respectively. Death of the matures for each population is modelled by quadratic terms, as in the logistic equation.

In the model (1.1) it is assumed that competition effects are of the classical Lotka–Volterra kind, and the effects of $v$ on $u$, and $u$ on $v$ are measured by $p_1$ and $p_2$, respectively. For the sake of simplicity we have dropped the uncoupled equations that describe the dynamics of the immature individuals of the populations $u$ and $v$. For a more detailed description of the model (1.1), the reader is referred to [3, 6]. For studies on delayed models related to (1.1), see [1, 2, 7].

Al-Omari and Gourley [3] studied the existence of travelling wave solutions in (1.1) in the case that

$$\int_{\mathbb{R}} G_u(y)u(t - \tau_1, x - y) dy = u(t - \tau_1, x), \quad \int_{\mathbb{R}} G_v(y)v(t - \tau_2, x - y) dy = v(t - \tau_2, x)$$  \hspace{1cm} (1.3)

where $G_u(y)$ and $G_v(y)$ are the standard Dirac function, which can be viewed as the limit of $G_{ui}(y)$ and $G_{vi}(y)$ given in (1.2) as $d_{ui} \to 0$ and $d_{vi} \to 0$. The hypothesis (1.3) implies that only the adult members of each population can diffuse. Al-Omari and Gourley showed that in this case there exists a positive number $\hat{c}$ implicitly depending on model parameters such that for $c \geq \hat{c}$ there exists a travelling wave solution with speed $c$ connecting two mono-culture equilibria under a technical assumption. (The notation $c^*$ instead of $\hat{c}$ was used in [3]. We shall reserve $c^*$ for the number given in (2.5).) Such travelling wave solutions describe the spatial spread of the adult individuals of the population with stronger competition ability into the adult individuals of its rival, and can provide insight about spatial patterns of competing species in space. The results in [3] represent significant progress in establishing travelling wave solutions for (1.1). However, the problem about the existence of travelling wave solutions for (1.1) has not been completely solved. In this paper we attempt to provide the sharpest theoretical result regarding the existence of travelling wave solutions connecting an unstable mono-culture equilibrium with a nontrival equilibrium in (1.1). We show that there is a positive number $c^*_+$ such that for $c \geq c^*_+$ the model (1.1) has a monotone travelling wave solutions with speed $c$ connecting two mono-culture equilibria or one mono-culture equilibrium with the coexistence equilibrium depending on model parameter values, and such a travelling wave solution does not exist if $c < c^*_+$. It was pointed out in [3] that the technical assumption for the existence of travelling wave solutions is probably not essential for the existence of travelling waves. Our results show that it is indeed unnecessary. Our results also indicate that the lower bound of speeds of travelling wave solutions given in [3] may not be the lowest.

There have been extensive studies about the existence of travelling wave solutions in delayed reaction–diffusion equations; see [5, 7, 9, 12, 17–19] and references cited therein. Wu and Zou [18] developed a monotone iteration scheme in establishing travelling wave solutions for monotone delayed systems, which was used by Al-Omari and Gourley in [3]. Li et al [9] made use of a cross-iteration scheme to obtain the existence of travelling wave solutions connecting the origin with the coexistence equilibria in delayed competitive systems.
and Zhao [12] developed the mathematical theory regarding the spreading speeds and travelling wave solutions for cooperative systems with delay. Their results have been successfully applied to several specific models [4, 12]. The theory developed by Liang and Zhao requires that the system considered has only two equilibria, and it cannot be used to obtain travelling wave solutions in (1.1) that connect two nontrivial equilibria because of the presence of the origin on the boundary of the relevant domain.

In this paper we study the existence of travelling wave solutions for delayed systems that may have more than two equilibria. We first define an important extended real number \( c^*_u \), and obtain the results where \( c^*_u \) is related to speeds of travelling wave solutions for recursions with delay. We then apply the general results to (1.1) to obtain that \( c^*_u \) is the slowest speed of travelling wave solutions connecting two nontrivial equilibria. This paper is organized as follows. In section 2, we define \( c^*_u \) and use it to establish the existence of travelling wave solutions for delayed recursions. In section 3, we devote to applying the results obtained in section 2 to the model (1.1). In section 4, we give some concluding remarks. The proofs of some useful lemmas are included in the appendix.

2. Existence of travelling waves for recursions

We shall use \( \mathcal{H} \) to denote the habitat where the species grow, interact and migrate. \( \mathcal{H} \) is either the real line (the continuous habitat) or the subset of the real line which consist of all integral multiples of positive mesh size \( h \) (a discrete habitat). Let \( \tau \) be a nonnegative real number. We shall use boldface Roman symbols like \( u(\theta, x) \) to denote \( k \)-vector-valued functions of the two variable \( \theta \) and \( x \), and boldface Greek letters to stand for \( k \)-vectors, which may be thought of as constant vector-valued functions. We define \( u \preceq v \) to mean that \( u^i(\theta, x) \geq v^i(\theta, x) \) for all \( i = 1, 2, \ldots, k, \theta \in [-\tau, 0] \) and \( x \in \mathcal{H} \), and \( u \gg v \) to mean that \( u^i(\theta, x) > v^i(\theta, x) \) for all \( i, \theta \) and \( x \). We also define \( \max\{u(\theta, x), v(\theta, x)\} \) to mean the vector-valued function whose \( i \)-th component at \( (\theta, x) \) is \( \max\{u^i(\theta, x), v^i(\theta, x)\} \). We use the notation \( \mathbf{0} \) for the constant vector all of whose components are 0.

Let \( C \) be the set of all bounded continuous functions from \([−\tau, 0] \times \mathcal{H} \) to \( \mathbb{R}^k \), \( \bar{C} \) be the set of all bounded continuous from \([−\tau, 0] \times \mathcal{H} \) to \( \mathbb{R}^k \), and \( \mathcal{X} \) be the set of bounded continuous functions from \( \mathcal{H} \) to \( \mathbb{R}^k \). If \( r \in \bar{C} \) with \( r \gg \mathbf{0} \), we define the set of continuous functions

\[
C_r := \{ u \in C : \mathbf{0} \leq u \leq r \}.
\]

Moreover, we define the metric function \( d(\cdot, \cdot) \) in \( \bar{C} \) by

\[
d(\phi, \psi) = \sum_{k=0}^{\infty} \max_{|\theta| \leq k, \theta \in [-\tau, 0]} |\phi(\theta, x) - \psi(\theta, x)|^2 \quad \forall \phi, \psi \in \bar{C}
\]

so that \((\bar{C}, d)\) is a metric space. The convergence of a sequence \( \phi_n \) to \( \phi \) with respect to this topology is equivalent to the uniform convergence of \( \phi_n \) to \( \phi \) on bounded subsets of \([−\tau, 0] \times \mathcal{H} \).

We study the following discrete-time recursion:

\[
u_{n+1} = Q[u_n], \quad n = 0, 1, 2, \ldots
\]

where \( u_n(\theta, x) = (u^1_n(\theta, x), u^2_n(\theta, x), \ldots, u^k_n(\theta, x)) \), \( \theta \in [-\tau, 0] \) and \( x \in \mathcal{H} \) represents the population densities of the populations of \( k \) species at time \( n \) and point \( x \) with time delay \( \tau \). The operator \( Q \) is said to be order preserving if \( u \preceq v \) implies that \( Q[u] \geq Q[v] \). A recursion (2.2) in which \( Q \) has this property is said to be cooperative. A function is said to be an equilibrium of \( Q \) if \( Q[w] = w \), so that if \( u_0 = w \) in the recursion (2.2), then \( u_n = w \) for all \( n \geq l \). We shall study the evolution of the solution \( u_0 \) of the recursion (2.2) from a \( u_0 \) near an unstable constant
equilibrium \( \theta \). By introducing the new variable \( \hat{u} = u - \theta \) if necessary, we shall assume the unstable equilibrium \( \theta \) from which the system moves away is the origin \( 0 \).

We define the translation and reflection operators

\[
T_r[v](\theta, x) = v(\theta, x - y), \quad R[v](\theta, x) = v(\theta, -x).
\]

A set \( D \subset \mathcal{C}_r \) is said to be \( T \)-invariant if \( T_r[D] = D \) for any \( y \in \mathcal{N} \).

We shall make the following hypotheses on \( Q \).

**Hypotheses 2.1.**

i. \( \mathcal{Q}[0] = 0 \), and there is a vector \( \beta(\theta) \in \mathcal{C} \) with \( \beta(\theta) \gg 0 \) such that \( \mathcal{Q}[\beta] = \beta \), and if \( u_0 \)


is any vector in \( \mathcal{C} \) with \( \beta(\theta) \gg u_0 \gg 0 \), then the vector-valued function \( u_n \) obtained from the recursion (2.2) converges to \( \beta(\theta) \) uniformly on \([−\tau, 0]\) as \( n \) approaches infinity.

ii. \( \mathcal{Q} \) is order preserving on nonnegative functions, so that if \( u \geq v \geq 0 \), then \( \mathcal{Q}[u] \geq \mathcal{Q}[v] \geq 0 \).

iii. \( \mathcal{Q} \) is translation and reflection invariant so that \( \mathcal{Q}[T_r[v]] = T_r[\mathcal{Q}[v]] \) for all \( y \) and \( \mathcal{Q}[R[v]] = R[\mathcal{Q}[v]] \).

iv. \( \mathcal{Q} \) is continuous with respect to the topology determined by \( d(\cdot, \cdot) \) given in (2.1).

v. One of the following two properties holds:

a. \( \mathcal{Q}[\mathcal{C}_\beta] \) is precompact in \( \mathcal{C}_\beta \).

b. The set \( \mathcal{Q}[\mathcal{C}_\beta](0, \cdot) \) is precompact in \( \mathcal{X} \), and there is a positive number \( \xi \leq \tau \) such that \( \mathcal{Q}[u](\theta, x) = u(\theta + \xi, x) \) for all \( \theta \in [−\tau, −\xi] \), and the operator

\[
S[u](\theta, x) = \begin{cases} 
(u(0, x), & \theta \in [−\tau, −\xi), \\
\mathcal{Q}[u](\theta, x), & \theta \in [−\xi, 0], 
\end{cases}
\]

has the property that \( S[D] \) is precompact in \( \mathcal{C}_\beta \) for any \( T \)-invariant set \( D \subset \mathcal{C}_\beta \) with \( D(0, \cdot) \) precompact in \( \mathcal{X} \).

**Remark 2.1.** Hypotheses 2.1 i–ii imply that \( \mathcal{Q} \) takes \( \mathcal{C}_\beta \) into itself, and that the equilibrium \( \beta \) attracts all initial functions in \( \mathcal{C}_\beta \) with uniformly positive components. In biological terms, \( \beta \) is a globally stable coexistence equilibrium. There may also be other equilibria lying between \( \beta \) and the extinction equilibrium \( 0 \), in each of which at least one of the species is extinct. Throughout this paper, we shall assume that the recursion (2.2) has a finite number of equilibria and that the equilibria of (2.2) are completely separate in the sense that for any two equilibria \( \nu_1(\theta), \nu_2(\theta) \in \mathcal{C} \) of (2.2), if \( \nu_1(\theta) \neq \nu_2(\theta) \) for some \( \theta \in [−\tau, 0] \) then \( \nu_1(\theta) \neq \nu_2(\theta) \) for all \( \theta \in [−\tau, 0] \).

**Remark 2.2.** Liang and Zhao [12] imposed the hypothesis that for any positive number \( \epsilon \), there is \( \alpha \in \mathcal{C} \) with \( 0 \leq \alpha \leq \beta \) and \( \|\alpha\| \leq \epsilon \) such that \( \mathcal{Q}[\alpha] \gg \alpha \). We have replaced this hypothesis by the convergence hypothesis in hypothesis 2.1 i.

The following lemma is useful in our discussion, which can be found in Lui [8], Weinberger et al [15], and Liang and Zhao [12].

**Lemma 2.1 (Comparison lemma).** Let \( R \) be an order preserving operator. If \( u_n \) and \( v_n \) satisfy the inequalities \( u_n \leq R[u_n] \) and \( v_n \geq R[v_n] \) for all \( n \), and if \( u_0 \leq v_0 \), then \( u_n \leq v_n \) for all \( n \).

We choose a continuous vector-valued function \( \phi(\theta, x) = (\phi^1, ..., \phi^k) \in \mathcal{C}_\beta \) that has the properties

B1 \( \phi^i(\theta, x) \) is nonincreasing in \( x \) for any \( \theta \in [−\tau, 0] \) and \( 1 \leq i \leq k \);

B2 \( \phi^i(\theta, x) = 0 \) for any \( \theta \in [−\tau, 0] \), \( x \geq 0 \), and \( 1 \leq i \leq k \);
(B3) \( \phi(\theta, -\infty) = \alpha(\theta) \) for any \( \theta \in [-\tau, 0] \), where \( 0 < \alpha(\theta) \ll \beta \) and \( \alpha(\theta) \in \mathcal{C} \).

For any real number \( c \), we define the operator
\[
R_c[u](\theta, s) = \max\{\phi(\theta, s), T_{-c}[Q[u]](\theta, s)\}
\]
for \( u \in C_\beta \). The operator \( R_c \) is again order preserving, and it takes \( C_\beta \) into itself.

We now define a sequence of vector-valued functions \( a_n(c; \theta, s) \) of \( (\theta, s) \in [-\tau, 0] \times \mathcal{H} \) by the recursion
\[
a_{n+1}(c; \theta, s) = R_c[a_n(c; \cdot)](\theta, s), \quad a_0(c; \theta, s) = \phi(\theta, s).
\]
(2.3)

It is easily seen that \( a_1 \geq a_0 \), \( a_1(c; \theta, s) \) is nonincreasing in \( c \) and \( s \), and continuous in \( (c, \theta, s) \). Induction shows that \( a_n \leq a_{n+1} \leq \beta \) for all \( n \), and \( a_n(c; \theta, s) \) is nonincreasing in \( c \) and \( s \) and continuous in \( (c, \theta, s) \). Hypotheses 2.1 imply that \( \{a_n(c; \theta, s) : n \geq 1, c \in \mathbb{R}\} \) is a family of equicontinuous functions of \( (\theta; s) \) in any bounded subset of \( [-\tau, 0] \times \mathcal{H} \) (see the proof of theorem 4.2 in Liang and Zhao [12]). It follows that the sequence \( a_n(c; \theta, s) \) converges to a limit function \( \hat{a}(c; \theta, s) \) uniformly in any bounded subset of \( [-\tau, 0] \times \mathcal{H} \) which is again nonincreasing in \( c \) and \( s \) and continuous in \( (\theta, s) \), and bounded above by \( \beta \). It is shown in [12] that the vector-valued function \( \hat{a}(c; \theta, \infty) \) is an equilibrium of the operator \( Q \) (see lemma 2.10 in [12] whose proof is still valid under hypotheses 2.1). It is also shown in [12] that \( a(c; \theta, -\infty) \) is an equilibrium of the operator \( Q \). We prove that this is still true under hypotheses 2.1.

**Lemma 2.2.** For any real number \( c \), \( a(c; \theta, -\infty) = \beta \).

The proof of this lemma can be found in the appendix.

It is easily seen that when \( c \) is sufficiently negative, \( a(c; \theta, s) = \beta \), or equivalently that \( a(c; \theta, \infty) = \beta \). If we start with a different function \( \hat{\phi} \) with the properties (B1)–(B3), we obtain a different sequence \( \hat{a}_n(c; \theta, s) \) and a different limit function \( \hat{a}(c; \theta, s) \). The proof of lemma 2.8 in [12] still works to show that \( a(c; \theta, \infty) = \hat{a}(c; \theta, \infty) \), i.e.,

\[
a(c; \theta, \infty) \quad \text{is independent of the initial function } \phi(\theta, s).
\]

We now examine what happens when the extra assumption
\[
0 \quad \text{and } \beta \quad \text{are the only equilibria of } Q \text{ in } C_\beta
\]
(2.4)

made in [12] is dropped. We can still define the function \( a(c; \theta, s) \) as above, and follow Liang and Zhao [12] to define
\[
c^* := \sup\{c : a(c; \theta, \infty) = \beta\}.
\]
(2.5)

The only difference is that \( a(c^*; \theta, \infty) \) may be an equilibrium \( \nu(\theta) \) other than 0. We define a second number
\[
c^*_+ := \sup\{c : a(c; \theta, \infty) \neq 0\}.
\]
(2.6)

It is easily seen that \( c^* \leq c^*_+ \), and if the extra assumption (2.4) is satisfied then \( c^* = c^*_+ \).

Since \( a(c; \theta, \infty) \) is the limit of a nonincreasing family of continuous functions, it is lower semi-continuous in \( c \). By the properties of lower semi-continuous functions, we see that the following lemma is true.

**Lemma 2.3.** \( a(c; \theta, \infty) \neq 0 \) if and only if \( c < c^*_+ \).

We now relate \( c^*_+ \) to speeds of travelling wave solutions in (2.2). A travelling wave solution of (2.2) with speed \( c \) has the form \( u_n(c; \theta, x) = W(c; \theta, x - nc) \) with \( W(c; \theta, s) \) a function in \( C_\beta \). That is, the solution at \( n + 1 \) is simply the translation by \( c \) of its value at \( n \). We have following theorem on the existence of travelling waves in (2.2).
Theorem 2.1. Let $c^*_n$ be defined by (2.6). Suppose that the operator $Q$ satisfies hypotheses 2.1. Then the following statements are true for system (2.2).

i. If $c \geq c^*_n$, then there is a nonincreasing travelling wave solution $W(c; \theta, x - nc)$ of speed $c$ with $W(c; \theta, \infty) = 0$ and $W(c; \theta, -\infty) = 0$ and $W(c; \theta, -\infty) = \beta$, then $c \geq c^*_n$.

Proof. We begin with the proof of the first statement. Let $c \geq c^*_n$. We choose a fixed vector-valued initial function $\phi(\theta, s) \in C_\beta$ with properties (B1)–(B3). For any $\kappa \in (0, 1]$ we define an operator $R_{c, \kappa}$ by

$$R_{c, \kappa}[\phi](\theta, \kappa) := \max(\kappa \phi(\theta, s), T_{c, \kappa}[Q[\phi]](\theta, s)),$$

and a sequence of vector-valued functions $a_n(\kappa, \theta, \kappa) = \max(\kappa, \phi(\theta, s), Q[\phi](\theta, s))$. (2.7)

As shown in the proof of theorem 4.2 of Liang and Zhao [12], $[a_n(\kappa, \theta, \kappa) : n \geq 1, \kappa \in (0, 1)]$ is a family of equicontinuous functions of $(\theta, \kappa)$ in any bounded subset of $[−\tau, 0] \times \mathcal{H}$. It follows that there is a sequence $n$, such that $a_n(\kappa, \theta, \kappa)$ converges uniformly for $\theta \in [−\tau, 0]$ and $s$ on bounded sets. Since $a_n$ is nondecreasing in $n$, the whole sequence $a(\kappa, \theta, \kappa)$ converges to a function $a(\kappa, \theta, \kappa)$ uniformly for $\theta \in [−\tau, 0]$ and $s$ on bounded sets. In particular, $a(\kappa, \theta, s)$ is a continuous function of $(\theta, s)$. By hypothesis 2.1 iv, we may take limits in (2.7) to see that

$$a(\kappa, \theta, s) = \max(\kappa \phi(\theta, s), Q[a(\kappa, \theta, \kappa)](\theta, s)).$$

(2.8)

Fix $\theta_0 \in [−\tau, 0]$. We choose a positive integer $N$ so large that there is no equilibrium $u$ other than $0$ satisfying $|u(\theta_0, x)| \leq 2/|\beta(\theta_0)|/N$ where $|·|$ denotes the Euclidean norm.

We choose $x_0 \in \mathcal{H}$ such that $x_0 > 0$. For every integer $\ell$ we define

$$K_\ell(\ell) := \frac{1}{N}[a(c, \kappa, \theta_0, \ell x_0) + \cdots + a(c, \kappa, \theta_0, (\ell + N - 1) x_0)].$$

(2.9)

$K_\ell(\ell)$ is nonincreasing in $\ell$. $K_\ell(−\infty) = |\beta(\theta_0)|$ due to lemma 2.2, and $K_\ell(\infty) = 0$ because of $c \geq c^*_n$. Since $a(c, \kappa, \theta_0, s)$ decreases from $\beta(\theta_0)$ to $0$ as $s$ goes from $−\infty$ to $\infty$, $K_\ell(\ell + 1) - K_\ell(\ell) \leq \frac{1}{N}|a(c, \kappa, \theta_0, (\ell + 1)x_0) + \cdots + a(c, \kappa, \theta_0, (\ell + N - 1)x_0)|$

$$\leq \frac{1}{N}[a(c, \kappa, \theta_0, (\ell + N)x_0) - a(c, \kappa, \theta_0, \ell x_0)]$$

$$\leq \frac{1}{N}|\beta(\theta_0)|.$$

Consequently, there is an integer which we call $\ell_\kappa$ such that

$$\frac{1}{N}|\beta(\theta_0)| \leq K_\ell(\ell_\kappa) \leq \frac{2}{N}|\beta(\theta_0)|.$$

(2.10)

We now consider the sequence $a(c, \kappa, \theta_0, s + \ell_\kappa x_0)$. We claim that the sequence $a(c, \kappa, \theta_0, s)$ forms a family of equicontinuous functions of $(\theta_0, s)$ in any bounded subset of $[−\tau, 0] \times \mathcal{H}$. In fact, for any $\theta_1, \theta_2 \in [−\tau, 0]$ and real numbers $s_1$ and $s_2$, we have that for any positive integer $n$

$$|a(c, \kappa, \theta_1, s_1) - a(c, \kappa, \theta_2, s_2)| \leq |a(c, \kappa, \theta_1, s_1) - a_n(c, \kappa, \theta_1, s_1)|$$

$$+ |a(c, \kappa, \theta_2, s_2) - a_n(c, \kappa, \theta_2, s_2)| + |a_n(c, \kappa, \theta_1, s_1) - a_n(c, \kappa, \theta_2, s_2)|.$$
Since \( a_n \) increases to \( a \) uniformly on bounded sets and \([a_n(c, \kappa; \theta, x) : n \geq 0, \kappa \in (0, 1)]\) is a family of equicontinuous functions of \((\theta, s)\), we have that the above claim is true.

It follows that there is a sequence \( \kappa_i \to 0 \) such that \( a(c, \kappa_i; \theta, x + \ell \kappa_0) \) converges uniformly for \( \theta \in [-\tau, 0] \) and \( x \) on bounded sets to a function \( W(c; \theta, x) \) that is nonincreasing in \( x \). We may take limits in (2.8) with \( \kappa = \kappa_i, x = x + \ell \kappa_0 - c \) and use the translation invariance of \( Q \) to find that

\[
W(c; \theta, s - c) = Q[W(c; \cdot)](\theta, s).
\]

Therefore, \( u_n(\theta, x) = W(c; \theta, x - nc) \) is a travelling wave solution of the recursion (2.2).

Letting \( s \) approach \( \pm\infty \) in (2.11) shows that \( W(c; \theta, \pm\infty) \) are equilibria of \( Q \). The definition (2.9) shows that the sequence \( K_n(\ell \kappa_i) \) convergence to

\[
\frac{1}{N}[|W(c; \theta, 0)| + \cdots + |W(c; \theta, (N - 1)x_0)|].
\]

Since \( W(c; \theta, x) \) is nonincreasing in \( x \), it follows from this and (2.10) that

\[
|W(c; \theta_0, \infty)| \leq \frac{2}{N}|\beta(\theta_0)|
\]

and

\[
\frac{1}{N}|\beta(\theta_0)| \leq |W(c; \theta_0, -\infty)|.
\]

Since \( W(c; \theta, x) \) is nonincreasing in \( x \), we have \( W(c; \theta, -\infty) \neq 0 \), and \( W(c; \theta, \infty) = 0 \) due to the choice of \( N \). This completes the proof of the first statement of the theorem.

To prove the second statement of the theorem, we suppose that there is a nonincreasing travelling wave \( W(c; \theta, x - nc) \) with \( W(c; \theta, \infty) = 0 \) and \( W(c; \theta, -\infty) = \beta \). It follows from Dini’s theorem that

\[
\lim_{x \to -\infty} W(c; \theta, x) = \beta \text{ uniformly for } \theta \in [-\tau, 0].
\]

We thus may choose a function \( \phi(\theta, s) \) with the properties (B1)-(B3) and a real number \( b \) such that \( \phi(\theta, x) \leq W(c; \theta, x + b) \). Define a sequence \( a_n \) by the recursion (2.3) with \( a_0(c; \theta, x) = \phi(\theta, x) \). We have that \( Q[\phi(c; \cdot)](\theta, s + c) \leq Q[W(c; \cdot)](\theta, s + b + c) = W(c; \theta, s + b) \). Thus \( a_1(c; \theta, s) \leq W(c; \theta, s + b) \). Induction shows that \( a_n(c; \theta, x) \leq W(c; \theta, x + b) \), and thus the limit function \( a \) of \( a_n \) has the property that \( a(c; \theta, x) \leq W(c; \theta, x + b) \). It follows that \( a(c; \theta, \infty) \leq \lim_{x \to -\infty} W(c; \theta, x + b) = 0 \). By the definition of \( c^*_\ast, c \geq c^*_\ast \), the proof of the theorem is complete.

We have the following corollary of theorem 2.1, which is essentially theorem 4.2 in [12].

**Corollary 2.1.** Assume that \( Q \) satisfies hypotheses 2.1. Assume, in addition, that the extra assumption (2.4) is satisfied, i.e. (2.2) has only two equilibria \( 0 \) and \( \beta(\theta) \). Then the recursion (2.2) has a nonincreasing travelling wave solution \( W(c; \theta, x - nc) \) with speed \( c \) and the properties \( W(c; \theta, \infty) = 0 \) and \( W(c; \theta, -\infty) = \beta \) if and only if \( c \geq c^*_\ast \).

**Remark 2.3.** We can obtain the existence of travelling wave solutions with speeds related to \( c^\ast \) given by (2.5). In particular, one can show that \( c^\ast \) can be characterized as the slowest speed of a class of travelling waves in the sense that

i. if \( c \geq c^\ast \), there is a nonincreasing travelling wave solution \( W(c; \theta, x - nc) \) with \( W(c; \theta, -\infty) = \beta \) and \( W(c; \theta, \infty) \) an equilibrium other than \( \beta \); and

ii. if there is a nonincreasing travelling wave \( W(c; \theta, x - nc) \) with \( W(\theta, -\infty) = \beta \) such that for at least one component \( i \lim_{x \to -\infty} W_i(c; \theta, x) = 0 \) uniformly for \( \theta \in [-\tau, 0] \), then \( c \geq c^\ast \).
This result is an extension of theorem 3.1 in Li et al [11]. One can prove it by using the definition of \( c^* \) and the arguments similar to those in the proof of theorem 2.1 with replacing (2.10) by

\[
\frac{N - 2}{N} |\beta(\theta_0)| \leq K_x(\ell_v) \leq \frac{N - 1}{N} |\beta(\theta_0)|.
\]

In the rest of this section, we consider travelling waves for the continuous time semiflow \( \{Q_t\}_{t=0}^{\infty} \) on \( C_R \). \( W(c; \theta, x - ct) \) is said to be a travelling wave of \( \{Q_t\}_{t=0}^{\infty} \) if \( W : [-\tau, 0] \times R \rightarrow R^k \) and \( Q_t[W](c; \theta, x) = W(c; \theta, x - ct) \). It is shown in [11, 12] that the theorem 2.1 can be extended to the continuous time function \( u(\theta, t, x) \) by the following trick: one first applies this result to the recursion with the time-2^{-\ell} map \( Q_{2^{-\ell}} \) to obtain the existence result of travelling waves for times which are multiples of \( 2^{-\ell} \). As shown in the proof of theorem 4.4 in [12], one can take a limit as \( \ell \) approaches infinity to obtain the following continuous result of the existence of travelling waves.

**Theorem 2.2.** Suppose that for any \( t > 0, Q_t \) satisfies hypotheses 2.1. Let \( c^*_p \) be given by (2.6) where \( Q \) is replaced by \( Q_t \). Then the following statements are true for \( \{Q_t\}_{t=0}^{\infty} \).

i. If \( c \geq c^*_p \), then there is a nonincreasing travelling wave solution \( W(c; \theta, x - ct) \) of speed \( c \) with \( W(c; \theta, \infty) = 0 \) and \( W(c; \theta, -\infty) \) an equilibrium other than \( 0 \).

ii. If there is a nonincreasing travelling wave \( W(c; \theta, x - ct) \) with \( W(c; \theta, \infty) = 0 \) and \( W(c; \theta, -\infty) = \beta \), then \( c \geq c^*_p \).

Furthermore, if the extra assumption (2.4) is satisfied, then there is a nonincreasing travelling wave \( W(c; \theta, x - ct) \) with \( W(c; \theta, \infty) = 0 \) and \( W(c; \theta, -\infty) = \beta(\theta) \) if and only if \( c \geq c^*_p \).

The result on travelling wave solutions with speeds related to \( c^* \) described in remark 2.3 can also be extended to the continuous time semiflow \( \{Q_t\}_{t=0}^{\infty} \) on \( C_R \).

3. Existence of travelling waves in (1.1)

In this section we show the existence of travelling wave solutions in (1.1) using theorems 2.1 and 2.2 obtained in the preceding section. The model (1.1) has the trivial equilibrium \( E_0 = (0, 0) \), the mono-culture equilibria \( E_u = (u^*, 0) \) and \( E_v = (0, v^*) \) with

\[
\begin{align*}
u^* &= \frac{a_1}{\eta_1}, \\
v^* &= \frac{a_2}{\eta_2},
\end{align*}
\]

and the coexistence equilibrium \( E_+ = (u^*, v^*) \) where

\[
\begin{align*}
u^* &= \frac{a_2 p_1 - a_1 \eta_2}{p_1 p_2 - \eta_1 \eta_2}, \\
v^* &= \frac{a_1 p_2 - a_2 \eta_1}{p_1 p_2 - \eta_1 \eta_2}.
\end{align*}
\]

It is easily seen that \( E_+ \) exists if and only if \( a_2 p_1 < a_1 p_2 \) and \( a_1 p_2 < a_2 \eta_1 \) or \( a_2 p_1 > a_1 \eta_2 \) and \( a_1 p_2 > a_2 \eta_1 \). Al-Omari and Gourley [3] obtained global stability results for a nonspatial model that is a generalization of the nonspatial model corresponding to (1.1). Their results show that if

\[
a_1 p_2 < a_2 \eta_1,
\]

then \( E_u \) is unstable, \( E_+ \) exists and it is globally attracting if \( a_2 p_1 < a_1 \eta_2 \), and \( E_+ \) does not exist and \( E_v \) is globally attracting if \( a_2 p_1 > a_1 \eta_2 \). We shall assume that (3.1) is satisfied, and discuss the spatial transition from \( E_u \) to the target state

\[
\hat{E} := (\hat{u}, \hat{v}) = \begin{cases} E_+, & \text{if } a_2 p_1 < a_1 \eta_2, \\
E_v, & \text{if } a_2 p_1 > a_1 \eta_2. \end{cases}
\]
We first make a change of variables $u := u^* - u$ and $v := v$ to convert the competition system (1.1) into the cooperative system
\[
\begin{align*}
\frac{\partial u}{\partial t} &= d_u \frac{\partial^2 u}{\partial x^2} - \alpha_1 \int_{\mathbb{R}} G_u(y)(u^* - u(t - \tau_1, x - y)) \, dy + \eta_1 (u^* - u)^2 + p_1 (u^* - u)v, \\
\frac{\partial v}{\partial t} &= d_v \frac{\partial^2 v}{\partial x^2} + \alpha_2 \int_{\mathbb{R}} G_v(y)v(t - \tau_2, x - y) \, dy - \eta_2 v^2 - p_2 (u^* - u)v.
\end{align*}
\]
(3.3)

For this system,
\[
\beta = (u^* - \bar{u}, \bar{v}) = \begin{cases} 
(u^* - u^*, v^*), & \text{if } \alpha_2 p_1 < \alpha_1 \eta_2, \\
(u^*, v^*), & \text{if } \alpha_2 p_1 > \alpha_1 \eta_2.
\end{cases}
\]
(4.4)

Let $\tau = \max\{\tau_1, \tau_2\}$. $0$ and $\beta$ are the only two equilibria in $C_\beta$ when $\alpha_2 p_1 < \alpha_1 \eta_2$. There is an extra equilibrium $\nu = (u^*, 0)$ in $C_\beta$ when $\alpha_2 p_1 > \alpha_1 \eta_2$.

**Definition 3.1.** A function $(u(t, x), v(t, x)) : [-\tau, b) \times \mathbb{R} \rightarrow \mathbb{R}^2$, $b > 0$, with the properties that $(u, v)$ is $C^2$ in $x \in \mathbb{R}$ and $C^1$ in $t \in (0, b)$ is called a super-solution (sub-solution) of (3.3) on $[0, b)$ if for $t \in [0, b)$, $x \in \mathbb{R}$
\[
\begin{align*}
\frac{\partial u}{\partial t} &\geq (\leq) d_u \frac{\partial^2 u}{\partial x^2} - \alpha_1 \int_{\mathbb{R}} G_u(y)(u^* - u(t - \tau_1, x - y)) \, dy + \eta_1 (u^* - u)^2 + p_1 (u^* - u)v, \\
\frac{\partial v}{\partial t} &\geq (\leq) d_v \frac{\partial^2 v}{\partial x^2} + \alpha_2 \int_{\mathbb{R}} G_v(y)v(t - \tau_2, x - y) \, dy - \eta_2 v^2 - p_2 (u^* - u)v.
\end{align*}
\]

The following existence and comparison lemma for the model (3.3) is a consequence of applying the results in Martin and Smith [13].

**Lemma 3.1.** For any $(\phi, \psi) \in C_\beta$, system (3.3) has a unique classical solution $(u(t, x; \phi, \psi), v(t, x; \phi, \psi))$ for $(t, x) \in (\tau, \infty) \times \mathbb{R}$, where $(u(0, x; \phi, \psi), v(0, x; \phi, \psi)) = (\phi, \psi)$. Furthermore, for any pair of super-solution $(\bar{u}(t, x), \bar{v}(t, x))$ and sub-solution $(\tilde{u}(t, x), \tilde{v}(t, x))$ of (3.3) with $(0, 0) \leq (u(t, x), v(t, x)) \leq (\bar{u}(t, x), \bar{v}(t, x)) \leq (u^* - \bar{u}, \bar{v})$ for $t \in [-\tau, 0)$ and $x \in \mathbb{R}$, there holds $(0, 0) \leq (u(t, x), v(t, x)) \leq (\bar{u}(t, x), \bar{v}(t, x)) \leq (u^* - \bar{u}, \bar{v})$ for $t \geq 0$ and $x \in \mathbb{R}$.

The proof of this lemma will be given in the appendix.

This lemma together with the global stability results for (3.3) shows that the time $t$ solution map $Q_t$ of system (3.3) with $t > 0$ exists, and it satisfies hypotheses 2.1 i–ii. Hypothesis 2.1 iii is satisfied by $Q_t$, since (3.3) is an autonomous system and $G_u$ and $G_v$ are symmetric functions.

**Lemma 3.2.** For any $t > 0$, $Q_t$ satisfies hypothesis 2.1 iv with $\beta$ given by (3.4).

**Lemma 3.3.** For any $t > 0$, $Q_t$ satisfies hypothesis 2.1 v with $\beta$ given by (3.4).

The proofs of lemmas 3.2 and 3.3 will be given in the appendix.

Since (5.3) is an autonomous system, $(Q_t)_{t \geq 0}$ is a semiflow on $C_\beta$. We have that $Q_t$ satisfies all the conditions in theorem 2.2.

**Lemma 3.4.** Let $c^*_r$ be defined by (2.6) where $Q$ is replaced by the time one map $Q_1$ of (3.3), then $0 < c^*_r < \infty$.

The proof will be given in the appendix.

We now are in the position to prove our main theorem in this section.

**Theorem 3.1.** Assume that the condition (3.1) holds. Let $c^*_r$ be defined by (2.6) where $Q$ is the time one solution map $Q_1$ of (3.3). Then for $c \geq c^*_r$, the model (1.1) has a monotone travelling wave $W(x - ct)$ connecting $E_u$ to $E$, and such a travelling wave does not exists if $c < c^*_r$. 
Proof. It suffices to show that system (3.3) has a nonincreasing travelling wave solution \( W(x - ct) = (u(x - ct), v(x - ct)) \) with \( W(+\infty) = (0, 0) \) and \( W(-\infty) = \beta \) if only if \( c \geq c^*_* \). By theorem 2.2, we only need show that for any \( c \geq c^*_* \), there is no nonincreasing travelling wave solution in (3.3) that connects \( \theta = (0, 0) \) with \( \nu = (u^*, 0) \) in the case where \( \alpha_2 \beta > \alpha_1 \eta_2 \). Assume that in this case \((u(x - ct), v(x - ct))\) is a nonincreasing travelling wave solution in (3.3) with \( c \geq c^*_* \) connecting \( \theta = (0, 0) \) with \( \nu = (u^*, 0) \). It follows that \( v \equiv 0 \) and \( u(x - ct) \) is a nonincreasing travelling wave solution of

\[
\frac{\partial u}{\partial t} = d_u \frac{\partial^2 u}{\partial x^2} - \alpha_1 \int_{\mathbb{R}} G_u(y)(u^* - u(t - \tau, x - y)) \, dy + \eta_1 (u^* - u)^2.
\]

We therefore have \( \hat{u}(x - ct) = u^* - u(x - ct) \) is a nondecreasing travelling wave solution of

\[
\frac{\partial \hat{u}}{\partial t} = d_u \frac{\partial^2 \hat{u}}{\partial x^2} + \alpha_1 \int_{\mathbb{R}} G_u(y)\hat{u}(t - \tau, x - y) \, dy - \eta_1 \hat{u}^2 \tag{3.5}
\]

with \( \hat{u}(-\infty) = 0 \) and \( \hat{u}(+\infty) = u^* \). According to lemma 3.4, \( c^*_* > 0 \), so that \( c > 0 \). We therefore have that \( \hat{u}(x - ct) \) is a nondecreasing travelling wave solution of (3.5) with the speed \(-c < 0\) connecting the unstable equilibrium 0 with the stable equilibrium \( u^* \).

On the other hand, since the right-hand side of (3.5) becomes bigger when the only nonlinear term \(-\eta_1 \hat{u}^2\) is dropped, we have that the time \( t \) solution map of (3.5) is dominated by that of the linearized model. One can then use theorem 3.10 in Liang and Zhao [12] and the argument given in Fang et al [4, pp 2758–9] to show that the minimal travelling wave speed for (3.5) is given by \( \inf_{\mu > 0} \lambda(\mu)/\mu \) where \( \lambda(\mu) \) is the principle eigenvalue of the characteristic equation

\[
\lambda - d_u \mu^2 - \alpha_1 \int_{-\infty}^{\infty} G_u(y)e^{\mu y - \eta \lambda} \, dy = 0
\]

or equivalently

\[
\lambda - d_u \mu^2 - \alpha_1 \int_{-\infty}^{\infty} e^{\mu y - \eta \lambda} \, dy = 0.
\]

Since \( \lambda(0) > 0 \) and \( \lambda(\mu) \geq d_u \mu^2 \) for \( \mu > 0 \), one can easily see that \( \inf_{\mu > 0} \lambda(\mu)/\mu > 0 \). We therefore have that the speeds of nondecreasing travelling wave solutions of (3.5) connecting 0 and \( u^* \) are all positive. We obtain a contradiction and complete the proof of this theorem.

Remark 3.1. While the analysis carried out in this section and in the appendix related to (1.1) is for the case \( d_{1(u)} > 0 \) and \( d_{1(v)} > 0 \), one can easily see that it is still valid for the limiting case \( d_{1(u)} = d_{1(v)} = 0 \). We therefore have that theorem 3.1 is valid for (1.1) when \( G_u \) and \( G_v \) satisfy (1.3).

4. Concluding remarks

We established the existence of travelling wave solutions for the delayed competition model (1.1). We showed that for \( c \geq c^*_* \) with \( c^*_* \) given by (2.6) where \( Q \) is the time one solution map of the delayed system (3.3), there exists a travelling wave solution for the model that connects an unstable mono-culture equilibrium with the other mono-culture equilibrium or with the coexistence equilibrium depending on the model parameter values. We first developed the mathematical theory regarding the existence of travelling wave solutions for cooperative recursions with delay that are allowed to have more than two equilibria, and then applied the general theory to obtain travelling wave solutions for the model (1.1). The general theory provided in section 2 can be applied a large class of delayed cooperative reaction–diffusion equations. Theorems 2.1 and 2.2 given in section 2 are new even for cooperative systems without delay.
The issue on spreading speeds for (2.2) was not addressed in this paper. It was shown in Weinberger et al. [11, 15, 16] that for a cooperative multi-species system without delay that has more than two equilibria, different components may spread at different spreading speeds. One can construct concrete examples with delay based on the examples given in Weinberger et al. [11, 15, 16] to show that this is also the case for delayed systems. It would be of interest to study the problem of multiple spreading speeds and investigate how $c^+$ and $c^*$ are related to spreading speeds in delayed cooperative systems.

It is difficult to compute $c^n_+$ for the model (1.1) explicitly or implicitly. It was shown in Lewis et al. [10] that when $t_1 = t_2 = 0$, $c^n_+ = c^*$ with $c^*$ given by (2.5) where $Q$ is the time one solution map of the delayed system (3.3). It was also shown in [10] that in this case both $c^n_+$ and $c^*$ represent the asymptotic spreading speed at which the population with stronger competition ability spreads and the population with weaker competition ability retreats in space when local invasion of the population with stronger competition ability occurs in an environment where the weaker competitor has established an equilibrium distribution. It would be interesting to study if these results are still valid when $t_1 > 0$ and $t_2 > 0$.

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Appendix

**Proof of lemma 2.2.** As shown in the proof of theorem 4.2 of Liang and Zhao [12], $[a_n(c; \theta, s) : n \geq 1]$ is a family of equicontinuous functions of $(\theta, s)$ in any bounded subset of $[−r, 0] \times \mathcal{H}$. For each $n \geq 1$, $a_n(c; \theta, x)$ increases to $a_n(c; \theta, −\infty)$ as $x \to −\infty$. It follows that for each $n \geq 1$, there exists a sequence $k$ with $k \to \infty$ such that

$$u_k(\theta, x) := a_n(c; \theta, x - k + c)$$

converges to $a_n(c; \theta, −\infty)$ uniformly in any bounded subset of $[−r, 0] \times \mathcal{H}$. By hypothesis 2.1 iv,

$$\lim_{k \to \infty} Q[a_n(c; \theta, x - k + c)](0) = Q[a_n(c; \theta, −\infty)]$$

and thus

$$a_{n+1}(c; \theta, −\infty) = \max \{\phi(\theta, −\infty), Q[a_n(c; \theta, −\infty)]\}.$$

The last equation and an induction argument show that $a_n(c; \theta, −\infty) \geq Q^n[\alpha_0]$ where $\alpha_0 = \alpha(−\infty)$. Recall that $\beta \gg \alpha(−\infty) \gg 0$. Since $\lim_{n \to \infty} Q^n[\alpha_0] = \beta(\theta)$ according to hypothesis 2.1 i and $a_n(c; \theta, −\infty) \leq a(c; \theta, −\infty) \leq \beta(\theta)$, a squeezing argument shows that $a(c; \theta, −\infty) = \beta(\theta)$. The proof is complete.

**Proof of lemma 3.1.** Define $f_i : \mathcal{C} \times \mathcal{C} \to \mathcal{X}$, $i = 1, 2$, by

$$f_1(\phi, \psi)(x) = −\alpha_1 \int_{\mathbb{R}} G_u(y)(u^* - \phi(−\tau_1, x - y)) \, dy + \eta_1(u^* - \phi(0, x))^2 + p_1(u^* - \phi(0, x))\psi(0, x),$$

$$f_2(\phi, \psi)(x) = \alpha_2 \int_{\mathbb{R}} G_v(y)\psi(−\tau_2, x - y) \, dy - \eta_2\psi(0, x)^2 - p_2(u^* - \phi(0, x))\psi(0, x).$$  (5.1)
We can then rewrite (3.3) as follows:
\[
\begin{aligned}
d\frac{\partial u}{\partial t} &= d_u \frac{\partial^2 u}{\partial x^2} + f_1(u_t, v_t)(x), \\
d\frac{\partial v}{\partial t} &= d_v \frac{\partial^2 v}{\partial x^2} + f_2(u_t, v_t)(x),
\end{aligned}
\]  
(5.2)

where \(u_t, v_t \in \mathcal{C}\) with \(u_t(\theta, x) = u(t + \theta, x)\) and \(v_t(\theta, x) = v(t + \theta, x)\) for \(\theta \in [-\tau, 0], x \in \mathbb{R}\). Let \([T_u(t)]_{t \geq 0}\) and \([T_v(t)]_{t \geq 0}\) be the solution semigroup on \(X\) generated by the heat equations \(u_j = d_u \Delta u\) and \(v_j = d_v \Delta u\). We thus can write (5.2) into the following integral equations:
\[
\begin{aligned}
u(t, x) &= T_u(t)u(0, \cdot)(x) + \int_0^t T_u(t - s)f_1(u_s, v_s)(x) \, ds, \\
v(t, x) &= T_v(t)v(0, \cdot)(x) + \int_0^t T_v(t - s)f_2(u_s, v_s)(x) \, ds,
\end{aligned}
\]  
(5.3)

Under the abstract setting in [13], a mild solution of (5.2) is a solution to its associated integral equation (5.3). By the expressions of \(f_1\) and \(f_2\) one can easily check that \(f_1\) and \(f_2\) are Lipschitz continuous on any bounded subset of \(\mathcal{C} \times \mathcal{C}\). Let \(\mathcal{Z} = BUC(\mathbb{R}, \mathbb{R}^2)\) be the Banach space of all bounded and uniformly continuous functions from \(\mathbb{R}\) into \(\mathbb{R}^2\) with the usual supremum norm. Let \(\mathcal{Z}^+ = \{ (\phi_1, \phi_2) : (\phi_1, \phi_2) \in \mathcal{Z}, \phi(x) \geq 0, i = 1, 2 \}\). We claim that \(f_1\) and \(f_2\) are quasi-monotone on \(\mathcal{C}\) in the sense that
\[
\begin{aligned}
\lim_{h \to 0^+} \frac{1}{h} \text{dist}(\phi_2(0) - \phi_1(0) + h[f_1(\phi_2, \psi_2) - f_1(\phi_1, \psi_1)], \mathcal{Z}^+) = 0, \\
\lim_{h \to 0^+} \frac{1}{h} \text{dist}(\psi_2(0) - \psi_1(0) + h[f_2(\phi_2, \psi_2) - f_2(\phi_1, \psi_1)], \mathcal{Z}^+) = 0,
\end{aligned}
\]  
(5.4)

for all \(\phi_j, \psi_j \in \mathcal{C}_0\) for \(j = 1, 2\) with \((\phi_2, \psi_2) \geq (\phi_1, \psi_1)\). From the definitions of \(f_1\) and \(f_2\) in (5.1) we see that
\[
f_1(\phi_2, \psi_2) - f_1(\phi_1, \psi_1)
= -\alpha_1 \int_{\mathbb{R}} G_\alpha(y)(u^* - \phi_2(\tau_1, x, y)) \, dy + \eta_1(u^* - \phi_2(0, x))^2 \\
+ p_1(u^* - \phi_1(0, x))\psi_2(0, x) + \alpha_1 \int_{\mathbb{R}} G_\alpha(y)(u^* - \phi_1(\tau_1, x, y)) \, dy \\
- \eta_1(u^* - \phi_1(0, x))^2 - p_1(u^* - \phi_1(0, x))\psi_1(0, x)
\]

\[= \alpha_1 \int_{\mathbb{R}} G_\alpha(y)(\phi_2(\tau_1, x, y) - \phi_1(\tau, x, y)) \, dy \\
+ p_1(\psi_2(0, x) - \psi_1(0, x))(u^* - \phi_1(0, x)) \\
+ (\phi_1(0, x) - \phi_2(0, x))\{p_1\psi_2(0, x) + \eta_1(2u^* - \phi_1(0, x) - \phi_2(0, x))\}
\]

and
\[
f_2(\phi_2, \psi_2) - f_2(\phi_1, \psi_1)
= \alpha_2 \int_{\mathbb{R}} G_\alpha(y)\psi_2(\tau_2, x, y) \, dy - \eta_2\psi_2(0, x)^2 - p_2(u^* - \phi_2(0, x))\psi_2(0, x) \\
- \alpha_2 \int_{\mathbb{R}} G_\alpha(y)\psi_1(\tau_2, x, y) \, dy + \eta_2\psi_1(0, x)^2 + p_2(u^* - \phi_1(0, x))\psi_1(0, x)
\]

\[= \alpha_2 \int_{\mathbb{R}} G_\alpha(y)(\psi_2(\tau_2, x, y) - \psi_1(\tau_2, x, y)) \, dy \\
+ p_2(\phi_2(0, x)\psi_2(0, x) - \phi_1(0, x)\psi_1(0, x)) \\
+ (\psi_1(0, x) - \psi_2(0, x))\{\eta_2(\psi_1(0, x) + \psi_2(0, x)) + p_2u^*\}.
\]
It follows that for sufficiently small \( h > 0 \),
\[
\phi_2(0, x) - \phi_1(0, x) + h[B_2(\phi_2, \psi_2) - B_2(\phi_1, \psi_1)]
\]
\[
= h\alpha_1 \int_{\mathbb{R}} G_\epsilon(y)(\phi_2(-\epsilon, x - y) - \phi_1(-\epsilon, x - y)) \, dy
\]
\[
+ h\beta_1(\psi_2(0, x) - \psi_1(0, x))(u^* - \phi_1(0, x))
\]
\[
+ (\phi_2(0, x) - \phi_1(0, x))[1 - h(\eta_1(u^* - \phi_1(0, x) - \phi_2(0, x)) + p_1\psi_2(0, x))]
\]
\[\geq 0\]

and
\[
\psi_2(0, x) - \psi_1(0, x) + h[f_2(\phi_2, \psi_2) - f_2(\phi_1, \psi_1)]
\]
\[
= h\alpha_2 \int_{\mathbb{R}} G_\epsilon(y)(\psi_2(-\epsilon, x - y) - \psi_1(-\epsilon, x - y)) \, dy
\]
\[
+ h\beta_2(\phi_2(0, x)\psi_2(0, x) - \phi_1(0, x)\psi_1(0, x))
\]
\[
+ (\psi_2(0, x) - \psi_1(0, x))[1 - h(\eta_2(\psi_1(0, x) + \psi_2(0, x)) + p_2u^*)]
\]
\[\geq 0\]

It follows that (5.4) holds. Then the existence and uniqueness of \((u(t, x; \phi, \psi), v(t, x; \phi, \psi))\) follows from corollary 5 in [13] with \((S_1(t, s), S_2(t, s)) = (T_1(t, s), T_2(t, s)) = (T_1(t - s), T_2(t - s)), t \geq s \geq 0, (B_1(t, \phi, \psi), B_2(t, \phi, \psi)) = (f_1(\phi, \psi), f_2(\phi, \psi)), and v^* = (u^* - \tilde{u}, \tilde{v}), v^- = (0, 0)\). Moreover, by the semigroup theory given in the proof of theorem 1 in [13], it follows that \((u(t, x; \phi), v(t, x; \psi))\) is a classical solution for \( t > \tau \).

For simplicity, let \( \psi(\theta, x) = (\tilde{u}(\theta, x), \tilde{v}(\theta, x))\), \( \phi(\theta, x) = (\bar{u}(\theta, x), \bar{v}(\theta, x))\), \( \theta \in [-\pi, 0], x \in \mathbb{R} \). Then \( (0, 0) \leq \phi \leq \psi \leq (u^* - \tilde{u}, \tilde{v}) \) with \( \phi \leq \psi \) in \( \mathbb{C}_\beta \). Again by corollary 5 in [13], we have for \( t \geq 0, x \in \mathbb{R} \)
\[
(0, 0) \leq (u(t, x, \phi), v(t, x, \phi)) \leq (u(t, x, \psi), v(t, x, \psi)) \leq (u^* - \tilde{u}, \tilde{v}). \quad (5.5)
\]

By applying corollary 5 in [13] with \( v^* = (u^* - \tilde{u}, \tilde{v}) \) and \( v^- = (\bar{u}(t, x), \bar{v}(t, x)) \) and \( v^- = (0, 0) \), respectively, we obtain
\[
(\bar{u}(t, x), \bar{v}(t, x)) \leq (u(t, x, \phi), v(t, x, \phi)) \leq (u^* - \tilde{u}, \tilde{v}), \quad t \geq 0, x \in \mathbb{R} \quad (5.6)
\]
and
\[
(0, 0) \leq (u(t, x, \psi), v(t, x, \psi)) \leq (\tilde{u}(t, x), \tilde{v}(t, x)), \quad t \geq 0, x \in \mathbb{R} \quad (5.7)
\]
Combing (5.5)–(5.7), we have \((\tilde{u}(t, x), \tilde{v}(t, x)) \geq (\bar{u}(t, x), \bar{v}(t, x))\) for all \( t \geq 0, x \in \mathbb{R} \). This completes the proof.

**Proof of lemma 3.2.** Let \( \Phi, \Phi_1, \Phi_2 \in C_\beta \). For any \( \varepsilon > 0 \) and \( t_0 > 0 \), we define
\[
H(t, x) := |u(t, x; \Phi_1) - u(t, x; \Phi_2)| + |v(t, x; \Phi_1) - v(t, x; \Phi_2)|;
\]
\[
K := \sup_{t \in [0, t_0], x \in \mathbb{R}} H(t, x);
\]
\[
\Omega_r(z) := [-\tau, 0] \times [z - r, z + r], \quad \forall r > 0, \quad z \in \mathbb{R};
\]
\[
|\Phi|_{\mathbb{C}_\beta} := \sup_{(\theta, x) \in \Omega_\varepsilon} |\Phi(\theta, x)|;
\]
\[
\varepsilon_0 := \frac{1}{2(3 + \Delta)\Delta e^{\Delta t_0}};
\]
where \( \Delta := \alpha_1 + \alpha_2 + 2\eta_1u^* + 2\eta_2v^* + (p_1 + p_2)(2u^* + v^*) \). Without loss of generality, we assume \( K \geq \sup_{y \in [-\tau, 0], x \in \mathbb{R}} H(\theta, x) \). Then, there exists \((t^*, x^*) \in [0, t_0] \times \mathbb{R} \) such that
$H_1(\theta, x) \leq H(t^*, x^*) + \epsilon_0$ for $(t, \theta, x) \in [0, t_0] \times [-\tau, 0] \times \mathbb{R}$. We choose $\sigma = \epsilon / \omega^{\Delta n}$ and $M = M(\epsilon, t_0) > 0$ such that for any $t \in [0, t_0]$,

\[
\int_{\|y\| > M} \frac{1}{\sqrt{4\pi d_B t}} e^{-\frac{y^2}{4d_B t}} dy \leq \frac{\epsilon_0}{\mu^*},
\int_{\|y\| > M} \frac{1}{\sqrt{4\pi d_B t}} e^{-\frac{y^2}{4d_B t}} dy \leq \frac{\epsilon_0}{\mu^*},
\int_\mathbb{R} \frac{1}{\sqrt{4\pi d_B t}} e^{-\frac{\|y-H_1(0, x^*-y)\|^2}{4d_B t}} dy \leq |H_1|_{\Omega^{(x^*)}} + \epsilon_0,
\int_\mathbb{R} \frac{1}{\sqrt{4\pi d_B t}} e^{-\frac{\|y-H_1(0, x^*-y)\|^2}{4d_B t}} dy \leq |H_1|_{\Omega^{(x^*)}} + \epsilon_0,
\int_\mathbb{R} \int_\mathbb{R} \frac{1}{\sqrt{4\pi d_B t}} e^{-\frac{\|G_u(z)H_2(-\tau_1, x^*-y-z)\|^2}{4d_B t}} dy dz \leq |H_1|_{\Omega^{(x^*)}} + \epsilon_0,
\int_\mathbb{R} \int_\mathbb{R} \frac{1}{\sqrt{4\pi d_B t}} e^{-\frac{\|G_u(z)H_2(-\tau_1, x^*-y-z)\|^2}{4d_B t}} dy dz \leq |H_1|_{\Omega^{(x^*)}} + \epsilon_0.
\]

It follows from this, (5.3), $0 \leq u(t, x; \Phi_1) \leq u^* - \tilde{u} \leq u^*$, and $0 \leq v(t, x; \Phi_1) \leq \tilde{v} \leq v^*$ for $i = 1, 2$ that if $|\Phi_1(\theta, x) - \Phi_1(\theta, x)|_{\Omega^{(x^*)}} < \sigma$, then

\[
|u(t^*, x^*; \Phi_1) - u(t^*, x^*; \Phi_2)|
= \left| T_u(t^*)u_0(0, x^*; \Phi_1) + \int_0^{t^*} T_u(t^*-s) f_1((u_1(\cdot, \cdot; \Phi_1), v_1(\cdot, \cdot; \Phi_1))(x^*)) ds \right|
- T_u(t^*)u_0(0, x^*; \Phi_2) - \int_0^{t^*} T_u(t^*-s) f_1((u_1(\cdot, \cdot; \Phi_2), v_1(\cdot, \cdot; \Phi_2))(x^*)) ds
\leq T_u(t^*)|u_0(0, x^*; \Phi_1) - u_0(0, x^*; \Phi_2)|
+ \int_0^{t^*} T_u(t^*-s) |f_1((u_1(\cdot, \cdot; \Phi_1), v_1(\cdot, \cdot; \Phi_1)) - f_1((u_1(\cdot, \cdot; \Phi_2), v_1(\cdot, \cdot; \Phi_2)))(x^*)| ds
\leq T_u(t^*)H_2(0, x^*) + \alpha_1 \int_0^{t^*} T_u(t^*-s) \int_{-\infty}^{\infty} G_u(z)H_2(-\tau_1, x-z) dz ds
+ 2\eta_1u^* \int_0^{t^*} T_u(t^*-s) H_2(0, x^*) ds + p_1(2u^* + v^*) \int_0^{t^*} T_u(t^*-s) H_2(0, x^*) ds
\leq \sigma + \varepsilon_0 + \alpha_1 \int_0^{t_0} (|H_2|_{\Omega^{(x^*)}} + \varepsilon_0) ds + 2\eta_1u^* \int_0^{t_0} (|H_2|_{\Omega^{(x^*)}} + \varepsilon_0) ds
+ p_1(2u^* + v^*) \int_0^{t_0} (|H_2|_{\Omega^{(x^*)}} + \varepsilon_0) ds
= \sigma + \varepsilon_0 (1 + \alpha_1 t_0 + 2\eta_1 u^* t_0 + p_1(2u^* + v^*) t_0)
+ (\alpha_1 + 2\eta_1 u^* + p_1(2u^* + v^*)) \int_0^{t_0} (|H_2|_{\Omega^{(x^*)}} + \varepsilon_0) ds.
\]

Similarly we can obtain that

\[
|v(t^*, x^*; \Phi_1) - v(t^*, x^*; \Phi_2)| \leq \sigma + \varepsilon_0 (1 + \alpha_2 t_0 + 2\eta_2 u^* t_0 + p_2(2u^* + v^*) t_0)
+ (\alpha_2 + 2\eta_1 v^* + p_2(2u^* + v^*)) \int_0^{t_0} (|H_2|_{\Omega^{(x^*)}} + \varepsilon_0) ds.
\]
We thus obtain
\[
|H_t|_{\Omega_{\Phi}(x^*)} \leq \varepsilon_0 + H(t^*, x^*)
= \varepsilon_0 + |u(t^*, x^*; \Phi_1) - u(t^*, x^*; \Phi_2)| + |v(t^*, x^*; \Phi_2) - v(t^*, x^*; \Phi_2)|
\leq 2\sigma + \varepsilon_0(3 + \Delta)t_0 + \Delta \int_0^t |H_s|_{\Omega_{\Phi}(x^*)} \, ds.
\]

It is following from Gronwall’s inequality that
\[
|H_t^n|_{\Omega_{\Phi}(x^*)} \leq (2\sigma + \varepsilon_0(3 + \Delta)t_0)e^{\Delta t}, \quad \forall t \in [0, t_0].
\]

We thus have obtained that for any \( \varepsilon > 0 \) which is small enough, and compact subset \( \mathcal{X} \subset [-t, 0] \times \mathbb{R} \), there exist \( \sigma > 0 \) and a compact set \( \Omega_{\Phi}(x^*) \) such that \( \mathcal{X} \subset \Omega_{\Phi}(x^*) \)
and
\[
|H_t|_{\mathcal{X}} \leq |H_t|_{\Omega_{\Phi}(x^*)} < \varepsilon \quad \text{for} \ t \in [0, t_0] \quad \text{and} \quad |\Phi_1 - \Phi_2|_{\Omega_{\Phi}(x^*)} < \sigma.
\]

This shows that \( Q_t \) is continuous in \( \Phi \) with respect to the compact open topology uniformly for \( t \in [0, t_0] \). Note that the metric space \( (C_\beta, d) \) is complete. By the triangle inequality and the continuity of \( Q_t \) in \( t \) from lemma 3.1, it then follows that \( Q_t(\phi) \) is continuous in \( (t; \Phi) \) with respect to the compact open topology. This completes the proof of lemma 3.2.

**Proof of lemma 3.3.** The proof of this lemma is similar to that of lemma 2.3 in Fang et al [4].

We give the proof here for completeness.

Let \( T(t) = (T_\alpha(t), T_\beta(t)) \), where \( T_\alpha(t) \) is the solution map of \( \frac{d\phi}{dt} = a_\alpha \frac{\phi}{\tau} \) and \( T_\beta(t) \) is the solution map of \( \frac{d\psi}{dt} = a_\beta \frac{\psi}{\tau} \). It follows that \( \{T_\alpha(t)\}_{t \geq 0} \) and \( \{T_\beta(t)\}_{t \geq 0} \) are linear semigroups on \( \mathcal{X} \) and \( T_\alpha(t) \) and \( T_\beta(t) \) are compact for each \( t > 0 \). Let \( Q_t = (Q_t^\alpha, Q_t^\beta) \) and \( (\phi, \psi) \in C_\beta \).

Given \( t_0 > \tau \), then
\[
Q_{t_0}^\alpha(\phi, \psi)(\theta, x) = u(t_0 + \theta, x; \phi, \psi)
= T_\alpha(t_0 + \theta)\phi(0, \cdot)(x) + \int_0^{t_0 + \theta} T_\alpha(t_0 + \theta - s)f_1(u_s, v_s)(x) \, ds,
\]
and
\[
Q_{t_0}^\beta(\phi, \psi)(\theta, x) = v(t_0 + \theta, x; \phi, \psi)
= T_\beta(t_0 + \theta)\phi(0, \cdot)(x) + \int_0^{t_0 + \theta} T_\beta(t_0 + \theta - s)f_2(u_s, v_s)(x) \, ds.
\]

By the properties of \( T_\alpha(t) \) and \( T_\beta(t) \) and the boundness of \( f_j \), we see that \( Q_t \) is compact for each \( t > \tau \). Thus \( Q_t \) satisfies hypotheses 2.1 v.a. when \( t > \tau \).

Given \( t_0 \in (0, \tau] \), we now show that \( Q_{t_0}^\beta \) satisfies hypotheses 2.1 v.b. with \( Q = Q_{t_0}^\alpha \).

To prove \( S[D] \) is precompact, it suffices to show that for any given compact interval \( I \subset \mathbb{R} \), \( u(t, x; \phi, \psi) \) and \( v(t, x; \phi, \psi) \) are equicontinuous in \( (t, x) \in [0, t_0] \times I \) for all \( (\phi, \psi) \in D \). We shall only show that \( u(t, x; \phi, \psi) \) is equicontinuous in \( (t, x) \in [0, t_0] \times I \) for all \( (\phi, \psi) \in D \), since the other statement can be proved in an analogous way. By the absolute continuity of integral, we have that for any \( \varepsilon > 0 \), there exists \( \delta_0 > 0 \) such that for any \( t \in (0, \delta_0] \),
\[
\left| \int_0^t T_u(t - s)f_1(u_s, v_s)(x) \, ds \right| < \frac{\varepsilon}{12}.
\]

On the other hand, since \( D(0, \cdot) \) is precompact in \( \mathcal{X} \), for the above interval \( I \), there exists \( \delta_1 > 0 \) such that \( |\phi(0, x_1) - \phi(0, x_2)| < \varepsilon/24 \) and \( |\psi(0, x_1) - \psi(0, x_2)| < \varepsilon/24 \) for all
We claim that choosing $M > 0$ properly so that
\[
\varepsilon/6 < \delta_0.
\]
We then obtain that
\[
|u(t_1, x_1; \phi, \psi) - u(t_2, x_2; \phi, \psi)| < \varepsilon/2.
\]
Hence we obtain that $|u(t_1, x_1; \phi, \psi) - u(t_2, x_2; \phi, \psi)| < \varepsilon/2$. Since $Q_t$ is compact when $t > \tau$, if follows that $u(t, x; \phi, \psi)$ is equicontinuous in $(t, x) \in [\delta_0, t_0] \times I$ for all $(\phi, \psi) \in D$. That is, for the above $\varepsilon$ and $I$, there exists $\delta_2 > 0$ so that $|u(t_1, x_1; \phi, \psi) - u(t_2, x_2; \phi, \psi)| < \varepsilon/2$ for all $(\phi, \psi) \in D$ if $t_1, t_2 \in [\delta_0, t_0]$ and $x_1, x_2 \in I$ with $|t_1 - t_2| + |x_1 - x_2| < \delta_2$. Let $\delta := \min(\delta_0, \delta_1, \delta_2)$, then we have for any $\varepsilon > 0$ and $I \in \mathbb{R}$, there exists $\delta > 0$ such that $|u(t_1, x_1; \phi, \psi) - u(t_2, x_2; \phi, \psi)| < \varepsilon$ for all $(\phi, \psi) \in D$ if $t_1, t_2 \in [0, t_0]$ and $x_1, x_2 \in I$ with $|t_1 - t_2| + |x_1 - x_2| < \delta$. This completes the proof of lemma 3.3.

**Proof of lemma 3.4.** We write the time one solution operator $Q_t$ of (3.3) as $Q_t = (Q_{u_1}, Q_{v_1})$. Let $\hat{Q}_{u_1}$ be the time 1 solution map of
\[
\frac{\partial v}{\partial t} = d_2 \frac{\partial^2 v}{\partial x^2} + \alpha_2 \int_{\mathbb{R}} G_{\epsilon}(y) v(t - t_2, x - y) \, dy - \eta_2 v^2 - p_2 u^2 v.
\]
The reaction term in (5.8) is $f_2(0, v_1)(x)$ where $f_2$ is given in (5.1). Since $f_2(u_1, v_1)(x) \geq f_2(0, v_1)(x)$, $Q_{u_1}(u, v) \geq \hat{Q}_{u_1}(v)$ for $(u, v) \in C_{\beta}$. The definition of $c^*_v$ and lemma 2.1 show that
\[
c^*_v \geq c^*_v
\]
where $c^*_v$ is given by (2.6) where $Q$ is $\hat{Q}_{u_1}$. Equation (5.8) has two constant equilibria $v = 0$ and $v = (\alpha_2 \eta_1 - \alpha_1 p_2)/(\eta_1 \eta_2) > 0$ due to (3.1). One can verify that under the condition (3.1), 0 is unstable and $(\alpha_2 \eta_1 - \alpha_1 p_2)/(\eta_1 \eta_2)$ is stable. Since the right-hand side of (5.8) becomes no smaller when the only nonlinear term $-\eta_2 v^2$ is dropped, we have that the time $t$ solution map of (5.8) is dominated by that of the
linearized model. One can then use theorem 3.10 in Liang and Zhao \cite{12} and the argument given in Fang et al \cite[pp 2758–9]{4} to find that
\[
c_{*} = \inf_{\mu>0} \lambda(\mu)/\mu,
\]
where \(\lambda(\mu)\) is the principal eigenvalue of the characteristic equation
\[
\eta_1 \lambda - \eta_1 d_1 \lambda^2 - \alpha_2 \eta_1 \int_{-\infty}^{\infty} G_\nu(y) e^{\delta y - \delta \lambda} \, dy + \alpha_1 p_2 = 0
\]
or equivalently
\[
\eta_1 \lambda - \eta_1 d_1 \lambda^2 - \alpha_2 \eta_1 e^{\delta d_1 \mu^2 - \delta \lambda^2} + \alpha_1 p_2 = 0.
\] (5.10)
The general results in section 3 of Liang and Zhao \cite{12} about estimates of spreading speeds indicate that \(\lambda(\mu) > 0\) for \(\mu \geq 0\). (One can directly verify that this is true using equation (5.10) and the condition (3.1)). On the other hand, \(\lambda(\mu) \geq \eta_1 d_1 \lambda^2 - \alpha_1 p_2\) for large positive \(\mu\) so that \(\lambda(\mu)/\mu \to \infty\) as \(\mu \to \infty\). It follows that the infimum of \(\lambda(\mu)/\mu\) occurs at a finite number and thus \(c_{*} = \inf\). We next show that \(c_{*} < 0\). We reorganize the first equation in (3.3) and then drop nonpositive terms from the resulting equation as well as from the second equation in (3.3) to obtain that the solutions of (3.3) satisfy
\[
\begin{align*}
\frac{\partial u}{\partial t} & \leq d_u \frac{\partial^2 u}{\partial x^2} + \alpha \int_R G_u(y) u(t - \tau_1, x - y) \, dy + p_i u^* v, \\
\frac{\partial v}{\partial t} & \leq d_v \frac{\partial^2 v}{\partial x^2} + \alpha \int_R G_v(y) v(t - \tau_2, x - y) \, dy.
\end{align*}
\] (5.11)
We shall show that there exist \(\lambda_0 > 0\) and \(c_0 > 0\) such that
\[
(u, v) = (e^{-\lambda_0(x-c_0 t)}, e^{-\lambda_0(x-c_0 t)})
\] (5.12)
satisfies
\[
\begin{align*}
\frac{\partial u}{\partial t} & \geq d_u \frac{\partial^2 u}{\partial x^2} + \alpha \int_R G_u(y) u(t - \tau_1, x - y) \, dy + p_i u^* v, \\
\frac{\partial v}{\partial t} & \geq d_v \frac{\partial^2 v}{\partial x^2} + \alpha \int_R G_v(y) v(t - \tau_2, x - y) \, dy.
\end{align*}
\] (5.13)
Substituting \((u, v)\) given by (5.12) into (5.13), we find that (5.13) is satisfied if and only if
\[
f(c_0, \lambda_0) > 0, \quad g(c_0, \lambda_0) > 0,
\]
where
\[
\begin{align*}
f(c, \lambda) & := c\lambda - d_u \lambda^2 - \alpha_1 \int_R G_u(y) e^{-\lambda(y-y_0)} \, dy - p_i u^*, \\
g(c, \lambda) & := c\lambda - d_v \lambda^2 - \alpha_2 \int_R G_v(y) e^{-\lambda(y-y_0)} \, dy.
\end{align*}
\]
It is easily seen that for any fixed \(\lambda > 0\)
\[
f(c, \lambda) > 0, \quad \lim_{c \to \infty} f(c, \lambda) = \infty \quad \text{and} \quad g(c, \lambda) > 0, \quad \lim_{c \to \infty} g(c, \lambda) = \infty.
\]
It follows that there exist \(\lambda_0 > 0\) and \(0 < c_0 < \infty\) such that \(f(c_0, \lambda_0) > 0\) and \(g(c_0, \lambda_0) > 0\). Since \((u, v) = (e^{-\lambda_0(x-c_0 t)}, e^{-\lambda_0(x-c_0 t)})\) satisfies (5.13) and solutions of (3.3) satisfy (5.11), \((u, v) = (e^{-\lambda_0(x-c_0 t)}, e^{-\lambda_0(x-c_0 t)})\) is an upper solution for (3.3). Let \(L_1\) be the time one solution map of the linear equation system corresponding to (5.13). We have that for \((u, v) \in C_j\), if \((u, v) \in (e^{-\theta_0}, e^{-\theta_0})\) with \(\theta \in [-\tau, 0]\) then
\[
T_{\tau_0} Q_i[u, v](x) \leq T_{-\tau} L_1[(e^{-\lambda_0}, e^{-\lambda_0})](x) \leq (e^{-\lambda_0}, e^{-\lambda_0}).
\] (5.14)
One can then use (2.3) to define the sequence \( a_n(c_0; \theta, x) \) with \( Q \) replaced by \( Q_1 \) and \( (\phi^1(\theta, x), \phi^2(\theta, x)) \) satisfying \( (\phi^1(\theta, x), \phi^2(\theta, x)) \leq (e^{-\lambda_0 x}, e^{-\lambda_0 x}) \) for \( \theta \in [-\tau, 0] \). Induction and (5.14) show that
\[
a_n(c_0; \theta, x) \leq (e^{-\lambda_0 x}, e^{-\lambda_0 x})
\]
for all \( n \) and thus the limit function 
\[
a(c_0; \theta, x) \leq (e^{-\lambda_0 x}, e^{-\lambda_0 x}).
\]
It follows immediately that 
\[
a(c_0; \theta, +\infty) = \lim_{x \to \infty} a(c_0; \theta, x) \leq \lim_{x \to \infty} (e^{-\lambda_0 x}, e^{-\lambda_0 x}) = (0, 0).
\]
The definition of \( c_0^* \) shows that \( c_0 \geq c_0^* \). This completes the proof of this lemma.

References