CHAPTER 3

Sequences

We begin our study of analysis with sequences. There are several reasons for starting here. First, sequences are the simplest way to introduce limits, the central idea of calculus. Second, sequences are a direct route to the topology of the real numbers. The combination of limits and topology provides the tools to finally prove the theorems you’ve already used in your calculus course.

1. Basic Properties

**Definition 3.1.** A sequence is a function \(a : \mathbb{N} \rightarrow \mathbb{R}\).

Instead of using the standard function notation of \(a(n)\) for sequences, it is usually more convenient to write the argument of the function as a subscript, \(a_n\).

**Example 3.1.** Let the sequence \(a_n = 1 - 1/n\). The first three elements are \(a_1 = 0, a_2 = 1/2, a_3 = 2/3\), etc.

**Example 3.2.** Let the sequence \(b_n = 2^n\). Then \(b_1 = 2, b_2 = 4, b_3 = 8\), etc.

**Example 3.3.** Let the sequence \(c_n = 100 - 5n\) so \(c_1 = 95, c_2 = 90, c_3 = 85\), etc.

**Example 3.4.** If \(a\) and \(r\) are constants, then a sequence given by \(c_1 = a, c_2 = ar, c_3 = ar^2\) and in general \(c_n = ar^{n-1}\) is called a geometric sequence. The number \(r\) is called the ratio of the sequence. Staying away from the trivial cases where \(a = 0\) or \(r = 0\), a geometric sequence can always be recognized by noticing that \(\frac{c_{n+1}}{c_n} = r\) for all \(n \in \mathbb{N}\). Example 3.2 is a geometric sequence with \(a = r = 2\).

**Example 3.5.** If \(a\) and \(d\) are constants, then a sequence of the form \(d_n = a + (n-1)d\) is called an arithmetic sequence. Another way of looking at this is that \(d_n\) is an arithmetic sequence if \(d_{n+1} - d_n = d\) for all \(n \in \mathbb{N}\). Example 3.3 is an arithmetic sequence with \(a = 95\) and \(d = -5\).

**Example 3.6.** Some sequences are not defined by an explicit formula, but are defined recursively. This is an inductive method of definition in which successive terms of the sequence are defined by using other terms of the sequence. The most famous of these is the Fibonacci sequence. To define the Fibonacci sequence, \(f_n\), let \(f_1 = 0, f_2 = 1\) and for \(n > 2\), let \(f_n = f_{n-2} + f_{n-1}\). The first few terms are \(0, 1, 1, 2, 3, 5, 8,\ldots\) There actually is a simple formula that directly gives \(f_n\), but we leave its derivation as Exercise 3.6.
Example 3.7. These simple definitions can lead to complex problems. One famous case is a hailstone sequence. Let \( h_1 \) be any natural number. For \( n > 1 \), recursively define
\[
h_n = \begin{cases} 
3h_{n-1} + 1, & \text{if } h_{n-1} \text{ is odd} \\
h_{n-1}/2, & \text{if } h_{n-1} \text{ is even}
\end{cases}
\]
Lothar Collatz conjectured in 1937 that any hailstone sequence eventually settles down to repeating the pattern 1, 4, 2, 1, 4, 2, \ldots. Many people have tried to prove this and all have failed.

It’s often inconvenient for the domain of a sequence to be \( \mathbb{N} \), as required by Definition 3.1. For example, the sequence beginning 1, 2, 4, 8, \ldots can be written \( 2^0, 2^1, 2^2, 2^3, \ldots \). Written this way, it’s natural to let the sequence function be \( 2^n \) with domain \( \omega \). As long as there is a simple substitution to write the sequence function in the form of Definition 3.1, there’s no reason to adhere to the letter of the law. In general, the domain of a sequence can be any set of the form \( \{ n \in \mathbb{Z} : n \geq N \} \) for some \( N \in \mathbb{Z} \).

Definition 3.2. A sequence \( a_n \) is bounded if \( \{ a_n : n \in \mathbb{N} \} \) is a bounded set. This definition is extended in the obvious way to bounded above and bounded below.

The sequence of Example 3.1 is bounded, but the sequence of Example 3.2 is not, although it is bounded below.

Definition 3.3. A sequence \( a_n \) converges to \( L \in \mathbb{R} \) if for all \( \varepsilon > 0 \) there exists an \( N \in \mathbb{N} \) such that whenever \( n \geq N \), then \( |a_n - L| < \varepsilon \). If a sequence does not converge, then it is said to diverge.

When \( a_n \) converges to \( L \), we write \( \lim_{n \to \infty} a_n = L \), or often, more simply, \( a_n \to L \).

Example 3.8. Let \( a_n = 1 - 1/n \) be as in Example 3.1. We claim \( a_n \to 1 \). To see this, let \( \varepsilon > 0 \) and choose \( N \in \mathbb{N} \) such that \( 1/N < \varepsilon \). Then, if \( n \geq N \)
\[
|a_n - 1| = |(1 - 1/n) - 1| = 1/n \leq 1/N < \varepsilon
\]
so \( a_n \to 1 \).

Example 3.9. The sequence \( b_n = 2^n \) of Example 3.2 diverges. To see this, suppose not. Then there is an \( L \in \mathbb{R} \) such that \( b_n \to L \). If \( \varepsilon = 1 \), there must be an \( N \in \mathbb{N} \) such that \( |b_n - L| < \varepsilon \) whenever \( n \geq N \). Choose \( n \geq N \). \( |L - 2^n| < 1 \) implies \( L < 2^n + 1 \). But, then
\[
b_{n+1} - L = 2^{n+1} - L > 2^{n+1} - (2^n + 1) = 2^n - 1 \geq 1 = \varepsilon.
\]
This violates the condition on \( N \). We conclude that for every \( L \in \mathbb{R} \) there exists an \( \varepsilon > 0 \) such that for no \( N \in \mathbb{N} \) is it true that whenever \( n \geq N \), then \( |b_n - L| < \varepsilon \). Therefore, \( b_n \) diverges.

Definition 3.4. A sequence \( a_n \) diverges to \( \infty \) if for every \( B > 0 \) there is an \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( a_n > B \). The sequence \( a_n \) is said to diverge to \( -\infty \) if \( -a_n \) diverges to \( \infty \).

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When \( a_n \) diverges to \( \infty \), we write \( \lim_{n \to \infty} a_n = \infty \), or often, more simply, \( a_n \to \infty \).

A common mistake is to forget that \( a_n \to \infty \) actually means the sequence diverges in a particular way. Don't be fooled by the suggestive notation into treating \( \infty \) as a number!

**Example 3.10.** It is easy to prove that the sequence \( a_n = 2^n \) of Example 3.2 diverges to \( \infty \).

**Theorem 3.5.** If \( a_n \to L \), then \( L \) is unique.

**Proof.** Suppose \( a_n \to L_1 \) and \( a_n \to L_2 \). Let \( \varepsilon > 0 \). According to Definition 3.2, there exist \( N_1, N_2 \in \mathbb{N} \) such that \( n \geq N_1 \) implies \( |a_n - L_1| < \varepsilon / 2 \) and \( n \geq N_2 \) implies \( |a_n - L_2| < \varepsilon / 2 \). Set \( N = \max\{N_1, N_2\} \). If \( n \geq N \), then

\[
|L_1 - L_2| = |L_1 - a_n + a_n - L_2| \leq |L_1 - a_n| + |a_n - L_2| < \varepsilon / 2 + \varepsilon / 2 = \varepsilon.
\]

Since \( \varepsilon \) is an arbitrary positive number an application of Exercise 2.12 shows \( L_1 = L_2 \).

**Theorem 3.6.** \( a_n \to L \) iff for all \( \varepsilon > 0 \), the set \( \{ n : a_n \not\in (L - \varepsilon, L + \varepsilon) \} \) is finite.

**Proof.** \((\Rightarrow)\) Let \( \varepsilon > 0 \). According to Definition 3.2, there is an \( N \in \mathbb{N} \) such that \( \{n : n \geq N\} \subset (L - \varepsilon, L + \varepsilon) \). Then \( \{n : a_n \not\in (L - \varepsilon, L + \varepsilon)\} \subset \{1, 2, \ldots, N - 1\} \), which is finite.

\((\Leftarrow)\) Let \( \varepsilon > 0 \). By assumption \( \{n : a_n \not\in (L - \varepsilon, L + \varepsilon)\} \) is finite, so let \( N = \max\{n : a_n \not\in (L - \varepsilon, L + \varepsilon)\} + 1 \). If \( n \geq N \), then \( a_n \in (L - \varepsilon, L + \varepsilon) \). By Definition 3.2, \( a_n \to L \).

**Corollary 3.7.** If \( a_n \) converges, then \( a_n \) is bounded.

**Proof.** Suppose \( a_n \to L \). According to Theorem 3.6 there are a finite number of terms of the sequence lying outside \((L - 1, L + 1)\). Since any finite set is bounded, the conclusion follows.

The converse of this theorem is not true. For example, \( a_n = (-1)^n \) is bounded, but does not converge. The main use of Corollary 3.7 is as a quick first check to see whether a sequence might converge. It's usually pretty easy to determine whether a sequence is bounded. If it isn't, it must diverge.

The following theorem lets us analyze some complicated sequences by breaking them down into combinations of simpler sequences.

**Theorem 3.8.** Let \( a_n \) and \( b_n \) be sequences such that \( a_n \to A \) and \( b_n \to B \). Then

(a) \( a_n + b_n \to A + B \),
(b) \( a_nb_n \to AB \), and
(c) \( a_n/b_n \to A/B \) as long as \( b_n \neq 0 \) for all \( n \in \mathbb{N} \) and \( B \neq 0 \).

**Proof.**

(a) Let \( \varepsilon > 0 \). There are \( N_1, N_2 \in \mathbb{N} \) such that \( n \geq N_1 \) implies \( |a_n - A| < \varepsilon / 2 \) and \( n \geq N_2 \) implies \( |b_n - B| < \varepsilon / 2 \). Define \( N = \max\{N_1, N_2\} \). If \( n \geq N \), then

\[
|(a_n + b_n) - (A + B)| \leq |a_n - A| + |b_n - B| < \varepsilon / 2 + \varepsilon / 2 = \varepsilon.
\]
Therefore $a_n + b_n \to A + B$.

(b) Let $\varepsilon > 0$ and $\alpha > 0$ be an upper bound for $|a_n|$. Choose $N_1, N_2 \in \mathbb{N}$ such that $n \geq N_1 \implies |a_n - A| < \varepsilon/2(|B| + 1)$ and $n \geq N_2 \implies |b_n - B| < \varepsilon/2\alpha$. If $n \geq N = \max\{N_1, N_2\}$, then
\[
|a_nb_n - AB| = |a_nb_n - a_nB + a_nB - AB| \\
\leq |a_nb_n - a_nB| + |a_nB - AB| \\
= |a_n||b_n - B| + |B||a_n - A| \\
< \alpha \frac{\varepsilon}{2\alpha} + |B|\frac{\varepsilon}{2(|B| + 1)} \\
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

(c) First, notice that it suffices to show that $1/b_n \to 1/B$, because part (b) of this theorem can be used to achieve the full result.

Let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that the following two conditions are satisfied: $n \geq N \implies |b_n| > |B|/2$ and $|b_n - B| < B^2\varepsilon/2$. Then, when $n \geq N$,
\[
\left| \frac{1}{b_n} - \frac{1}{B} \right| = \frac{B - b_n}{b_nB} < \frac{B^2\varepsilon/2}{(B/2)B} = \varepsilon.
\]

Therefore $1/b_n \to 1/B$.

If you’re not careful, you can easily read too much into the previous theorem and try to use its converse. Consider the sequences $a_n = (-1)^n$ and $b_n = -a_n$. Their sum, $a_n + b_n = 0$, product $a_nb_n = -1$ and quotient $a_n/b_n = -1$ all converge, but the original sequences diverge.

It is often easier to prove that a sequence converges by comparing it with a known sequence than it is to analyze it directly. For example, a sequence such as $a_n = \sin^2 n/n^3$ can easily be seen to converge to 0 because it is dominated by $1/n^3$.

The following theorem makes this idea more precise. It’s called the Sandwich Theorem here, but is also called the Squeeze, Pinching, Pliers or Comparison Theorem in different texts.

**Theorem 3.9 (Sandwich Theorem).** Suppose $a_n$, $b_n$ and $c_n$ are sequences such that $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$.

(a) If $a_n \to L$ and $c_n \to L$, then $b_n \to L$.

(b) If $b_n \to \infty$, then $c_n \to \infty$.

(c) If $b_n \to -\infty$, then $a_n \to -\infty$.

**Proof.**

(a) Let $\varepsilon > 0$. There is an $N \in \mathbb{N}$ large enough so that when $n \geq N$, then $L - \varepsilon < a_n$ and $c_n < L + \varepsilon$. These inequalities imply $L - \varepsilon < a_n \leq b_n \leq c_n < L + \varepsilon$. Theorem 3.6 shows $c_n \to L$.

(b) Let $B > 0$ and choose $N \in \mathbb{N}$ so that $n \geq N \implies b_n > B$. Then $c_n \geq b_n > B$ whenever $n \geq N$. This shows $c_n \to \infty$.

(c) This is essentially the same as part (b).
2. Monotone Sequences

One of the problems with using the definition of convergence to prove a given sequence converges is the limit of the sequence must be known in order to verify the sequence converges. This gives rise in the best cases to a “chicken and egg” problem of somehow determining the limit before you even know the sequence converges. In the worst case, there is no nice representation of the limit to use, so you don’t even have a “target” to shoot at. The next few sections are ultimately concerned with removing this deficiency from Definition 3.2, but some interesting side-issues are explored along the way.

Not surprisingly, we begin with the simplest case.

**Definition 3.10.** A sequence $a_n$ is **increasing**, if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$. It is **strictly increasing** if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.

A sequence $a_n$ is **decreasing**, if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$. It is **strictly decreasing** if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.

If $a_n$ is any of the four types listed above, then it is said to be a **monotone** sequence.

Notice the $\sum$ and $\prod$ in the definitions of increasing and decreasing sequences, respectively. Many calculus texts use strict inequalities because they seem to better match the intuitive idea of what an increasing or decreasing sequence should do. For us, the non-strict inequalities are more convenient.

**Theorem 3.11.** A bounded monotone sequence converges.

**Proof.** Suppose $a_n$ is a bounded increasing sequence, $L = \text{lub}\{a_n : n \in \mathbb{N}\}$ and $\varepsilon > 0$. Clearly, $a_n \leq L$ for all $n \in \mathbb{N}$. According to Theorem 2.19, there exists an $N \in \mathbb{N}$ such that $a_N > L - \varepsilon$. Because the sequence is increasing, $L \geq a_n \geq a_N > L - \varepsilon$ for all $n \geq N$. This shows $a_n \to L$.

If $a_n$ is decreasing, let $b_n = -a_n$ and apply the preceding argument. \qed

The key idea of this proof is the existence of the least upper bound of the sequence when the range of the sequence is viewed as a set of numbers. This means the Completeness Axiom implies Theorem 3.11. In fact, it isn’t hard to prove Theorem 3.11 also implies the Completeness Axiom, showing they are equivalent statements. Because of this, Theorem 3.11 is often used as the Completeness Axiom on $\mathbb{R}$ instead of the least upper bound property we used in Axiom 8.

**Example 3.11.** The sequence $e_n = (1 + \frac{1}{n})^n$ converges.

Looking at the first few terms of this sequence, $e_1 = 2$, $e_2 = 2.25$, $e_3 \approx 2.37$, $e_4 \approx 2.44$, it seems to be increasing. To show this is indeed the case, fix $n \in \mathbb{N}$ and use the binomial theorem to expand the product as

$$e_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}$$

(3.1)
and

\[ e_{n+1} = \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{1}{(n+1)^k}. \]  

For \(1 \leq k \leq n\), the \(k\)th term of (3.1) is

\[
\binom{n}{k} \frac{1}{n^k} = \frac{n(n-1)(n-2) \cdots (n-(k-1))}{k! n^k}
\]

\[
= \frac{1}{k!} \frac{n-1}{n} \frac{n-2}{n} \cdots \frac{n-(k-1)}{n}
\]

\[
< \frac{1}{k!} \left( \frac{n}{n+1} \right) \left( \frac{n-1}{n+1} \right) \cdots \left( \frac{n-(k-1)}{n+1} \right)
\]

\[
= \frac{1}{k!} \frac{n}{n+1} \frac{n-1}{n+1} \cdots \frac{n-(k-1)}{n+1}
\]

\[
= \frac{(n+1)n(n-1)(n-2) \cdots (n-(k-1))}{k!(n+1)^k}
\]

\[
= \binom{n+1}{k} \frac{1}{(n+1)^k},
\]

which is the \(k\)th term of (3.2). Since (3.2) also has one more positive term in the sum, it follows that \(e_n < e_{n+1}\), and the sequence \(e_n\) is increasing.

Noting that \(1/k! \leq 1/2^{k-1}\) for \(k \in \mathbb{N}\), we can bound the \(k\)th term of (3.1).

\[
\binom{n}{k} \frac{1}{n^k} = \frac{n!}{k!(n-k)!} \frac{1}{n^k}
\]

\[
= \frac{n-1}{n} \frac{n-2}{n} \cdots \frac{n-k+1}{n} \frac{1}{k!}
\]

\[
< \frac{1}{k!}
\]

\[
\leq \frac{1}{2^{k-1}}.
\]

Substituting this into (3.1) yields

\[
e_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k}
\]

\[
< 1 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{n-1}}
\]

\[
= 1 + \frac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} < 3,
\]

so \(e_n\) is bounded.
Since $e_n$ is increasing and bounded, Theorem 3.11 implies $e_n$ converges. Of course, you probably remember from your calculus course that $e_n \to \varepsilon \approx 2.71828$.

**Theorem 3.12.** An unbounded monotone sequence diverges to $\infty$ or $-\infty$, depending on whether it is increasing or decreasing, respectively.

**Proof.** Suppose $a_n$ is increasing and unbounded. If $B > 0$, the fact that $a_n$ is unbounded yields an $N \in \mathbb{N}$ such that $a_N > B$. Since $a_n$ is increasing, $a_n \geq a_N > B$ for all $n \geq N$. This shows $a_n \to \infty$.

The proof when the sequence decreases is similar. \qed

### 3. Subsequences and the Bolzano-Weierstrass Theorem

**Definition 3.13.** Let $a_n$ be a sequence and $\sigma : \mathbb{N} \to \mathbb{N}$ be a function such that $m < n$ implies $\sigma(m) < \sigma(n)$; i.e., $\sigma$ is a strictly increasing sequence of natural numbers. Then $b_n = a_{\sigma(n)}$ is a subsequence of $a_n$.

The idea here is that the subsequence $b_n$ is a new sequence formed from an old sequence $a_n$ by possibly leaving terms out of $a_n$. In other words, all the terms of $b_n$ must also appear in $a_n$, and they must appear in the same order.

**Example 3.12.** Let $\sigma(n) = 3n$ and $a_n$ be a sequence. Then the subsequence $a_{\sigma(n)}$ looks like

$$a_3, a_6, a_9, \ldots, a_{3n}, \ldots$$

The subsequence has every third term of the original sequence.

**Example 3.13.** If $a_n = \sin(n\pi/2)$, then some possible subsequences are

$$b_n = a_{4n+1} \implies b_n = 1,$$

$$c_n = a_{2n} \implies c_n = 0,$$

and

$$d_n = a_{n^2} \implies d_n = (1 + (-1)^{n+1})/2.$$  

**Theorem 3.14.** $a_n \to L$ iff every subsequence of $a_n$ converges to $L$.

**Proof.** ($\Rightarrow$) Suppose $\sigma : \mathbb{N} \to \mathbb{N}$ is strictly increasing, as in the preceding definition. With a simple induction argument, it can be seen that $\sigma(n) \geq n$ for all $n$. (See Exercise 3.8.)

Now, suppose $a_n \to L$ and $b_n = a_{\sigma(n)}$ is a subsequence of $a_n$. If $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $a_n \in (L - \varepsilon, L + \varepsilon)$. From the preceding paragraph, it follows that when $n \geq N$, then $b_n = a_{\sigma(n)} = a_m$ for some $m \geq n$. So, $b_n \in (L - \varepsilon, L + \varepsilon)$ and $b_n \to L$.

($\Leftarrow$) Since $a_n$ is a subsequence of itself, it is obvious that $a_n \to L$. \qed

The main use of Theorem 3.14 is not to show that sequences converge, but, rather to show they diverge. It gives two strategies for doing this: find two subsequences converging to different limits, or find a divergent subsequence. In Example 3.13, the subsequences $b_n$ and $c_n$ demonstrate the first strategy, while $d_n$ demonstrates the second.
Even if the original sequence diverges, it is possible there are convergent subsequences. For example, consider the divergent sequence \( a_n = (-1)^n \). In this case, \( a_n \) diverges, but the two subsequences \( a_{2n} = 1 \) and \( a_{2n+1} = -1 \) are constant sequences, so they converge.

**Theorem 3.15.** Every sequence has a monotone subsequence.

**Proof.** Let \( a_n \) be a sequence and \( T = \{ n \in \mathbb{N} : m > n \implies a_m \geq a_n \} \). There are two cases to consider, depending on whether \( T \) is finite.

First, assume \( T \) is infinite. Define \( \sigma(1) = \min T \) and assuming \( \sigma(n) \) is defined, set \( \sigma(n+1) = \min T \setminus \{ \sigma(1), \sigma(2), \ldots, \sigma(n) \} \). This inductively defines a strictly increasing function \( \sigma : \mathbb{N} \rightarrow \mathbb{N} \). The definition of \( T \) guarantees \( a_{\sigma(n)} \) is an increasing subsequence of \( a_n \).

Now, assume \( T \) is finite. Let \( \sigma(1) = \max T + 1 \). If \( \sigma(n) \) has been chosen for some \( n > \max T \), then the definition of \( T \) implies there is an \( m > \sigma(n) \) such that \( a_m \leq a_{\sigma(n)} \). Set \( \sigma(n + 1) = m \). This inductively defines the strictly increasing function \( \sigma : \mathbb{N} \rightarrow \mathbb{N} \) such that \( a_{\sigma(n)} \) is a decreasing subsequence of \( a_n \).

If the sequence in Theorem 3.15 is bounded, then the corresponding monotone subsequence is also bounded. Recalling Theorem 3.11, we arrive at the following famous theorem.

**Theorem 3.16 (Bolzano-Weierstrass).** Every bounded sequence has a convergent subsequence.

### 4. Lower and Upper Limits of a Sequence

There are an uncountable number of strictly increasing functions \( \sigma : \mathbb{N} \rightarrow \mathbb{N} \), so every sequence \( a_n \) has an uncountable number of subsequences. If \( a_n \) converges, then Theorem 3.14 shows all of these subsequences converge to the same limit. It’s also apparent that when \( a_n \to \infty \) or \( a_n \to -\infty \), then all its subsequences diverge in the same way. When \( a_n \) does not converge or diverge to \( \pm \infty \), the situation is a bit more difficult because some subsequences may converge and others may diverge.

**Example 3.14.** Let \( Q = \{ q_n : n \in \mathbb{N} \} \) and \( \alpha \in \mathbb{R} \). Since every interval contains an infinite number of rational numbers, it is possible to choose \( \sigma(1) = \min \{ k : |q_k - \alpha| < 1 \} \). In general, assuming \( \sigma(n) \) has been chosen, choose \( \sigma(n + 1) = \min \{ k > \sigma(n) : |q_k - \alpha| < 1/n \} \). Such a choice is always possible because \( Q \cap (\alpha - 1/n, \alpha + 1/n) \setminus \{ q_k : k \leq \sigma(n) \} \) is infinite. This induction yields a subsequence \( q_{\sigma(n)} \) of \( q_n \) converging to \( \alpha \).

If \( a_n \) is a sequence and \( b_n \) is a convergent subsequence of \( a_n \) with \( b_n \to L \), then \( L \) is called an accumulation point of \( a_n \). A convergent sequence has only one accumulation point, but a divergent sequence may have many accumulation points. As seen in Example 3.14, a sequence may have all of \( \mathbb{R} \) as its set of accumulation points.
To make some sense out of this, suppose \( a_n \) is a bounded sequence, and \( T_n = \{a_k : k \geq n\} \). Define

\[ \ell_n = \text{glb} \ T_n \quad \text{and} \quad \mu_n = \text{lub} \ T_n. \]

Because \( T_n \supset T_{n+1} \), it follows that for all \( n \in \mathbb{N} \),

\[ \ell_1 \leq \ell_n \leq \ell_{n+1} \leq \mu_{n+1} \leq \mu_n \leq \mu_1. \]  

(3.3)

This shows \( \ell_n \) is an increasing sequence bounded above by \( \mu_1 \) and \( \mu_n \) is a decreasing sequence bounded below by \( \ell_1 \). Theorem 3.11 implies both \( \ell_n \) and \( \mu_n \) converge. If \( \ell_n \to \ell \) and \( \mu_n \to \mu \), (3.3) shows for all \( n \),

\[ \ell \leq \ell_n \leq \mu \leq \mu_n. \]  

(3.4)

Suppose \( b_n \to \beta \) is any convergent subsequence of \( a_n \). From the definitions of \( \ell_n \) and \( \mu_n \), it is seen that \( \ell_n \geq b_n \geq \mu_n \) for all \( n \). Now (3.4) shows \( \ell \leq \beta \leq \mu \).

The normal terminology for \( \ell \) and \( \mu \) is given by the following definition.

**Definition 3.17.** Let \( a_n \) be a sequence. If \( a_n \) is bounded below, then the lower limit of \( a_n \) is

\[ \liminf a_n = \lim_{n \to \infty} \text{glb} \ \{a_k : k \geq n\}. \]

If \( a_n \) is bounded above, then the upper limit of \( a_n \) is

\[ \limsup a_n = \lim_{n \to \infty} \text{lub} \ \{a_k : k \geq n\}. \]

When \( a_n \) is unbounded, the lower and upper limits are set to appropriate infinite values, while recalling the familiar warnings about \( \infty \) not being a number.

**Example 3.15.** Define

\[ a_n = \begin{cases} 2 + 1/n, & n \text{ odd} \\ 1 - 1/n, & n \text{ even} \end{cases} \]

Then

\[ \mu_n = \text{lub} \ \{a_k : k \geq n\} = \begin{cases} 2 + 1/n, & n \text{ odd} \\ 2 + 1/(n+1), & n \text{ even} \end{cases} \]

and

\[ \ell_n = \text{glb} \ \{a_k : k \geq n\} = \begin{cases} 1 - 1/n, & n \text{ even} \\ 1 - 1/(n+1), & n \text{ even} \end{cases} \]

So,

\[ \limsup a_n = 2 > 1 = \liminf a_n. \]

Suppose \( a_n \) is bounded above and both \( \mu_n \) and \( \mu \) are as in the discussion preceding the definition. Choose \( \sigma(1) \) so \( a_{\sigma(1)} > \mu_1 - 1 \). If \( \sigma(n) \) has been chosen for some \( n \in \mathbb{N} \), then choose \( \sigma(n+1) > \sigma(n) \) to satisfy

\[ \mu_n \geq a_{\sigma(n+1)} > \text{lub} \ T_{n+1} - 1/n = u_{n+1} - 1/n. \]

This inductively defines a subsequence \( a_{\sigma(n)} \to \mu = \limsup a_n \), where the convergence is guaranteed by Theorem 3.9, the Sandwich Theorem.

In the cases when \( \limsup a_n = \infty \) and \( \limsup a_n = -\infty \), it is left to the reader to show there is a subsequence \( b_n \to \limsup a_n \).
Similar arguments can be made for \( \lim \inf a_n \).

To summarize: If \( \beta \) is an accumulation point of \( a_n \), then
\[
\lim \inf a_n \leq \beta \leq \lim \sup a_n.
\]

In case \( a_n \) is bounded, both \( \lim \inf a_n \) and \( \lim \sup a_n \) are accumulation points of \( a_n \) and \( a_n \) converges if \( \lim \inf a_n = \lim_{n \to \infty} a_n = \lim \sup a_n \).

The following theorem has been proved.

**Theorem 3.18.** Let \( a_n \) be a sequence.

(a) There are subsequences of \( a_n \) converging to \( \lim \inf a_n \) and \( \lim \sup a_n \).

(b) If \( \alpha \) is an accumulation point of \( a_n \), then \( \lim \inf a_n \leq \alpha \leq \lim \sup a_n \).

(c) \( \lim \inf a_n = \lim \sup a_n \in \mathbb{R} \) iff \( a_n \) converges.

### 5. The Nested Interval Theorem

**Definition 3.19.** A collection of sets \( \{S_n : n \in \mathbb{N}\} \) is said to be nested, if \( S_{n+1} \subseteq S_n \) for all \( n \in \mathbb{N} \).

**Theorem 3.20** (Nested Interval Theorem). If \( \{I_n = [a_n, b_n] : n \in \mathbb{N}\} \) is a nested collection of closed intervals such that \( \lim_{n \to \infty} (b_n - a_n) = 0 \), then there is an \( x \in \mathbb{R} \) such that \( \bigcap_{n \in \mathbb{N}} I_n = \{x\} \).

**Proof.** Since the intervals are nested, it’s clear that \( a_n \) is an increasing sequence bounded above by \( b_1 \) and \( b_n \) is a decreasing sequence bounded below by \( a_1 \). Applying Theorem 3.11 twice, we find there are \( \alpha, \beta \in \mathbb{R} \) such that \( a_n \to \alpha \) and \( b_n \to \beta \).

We claim \( \alpha = \beta \). To see this, let \( \varepsilon > 0 \) and use the “shrinking” condition on the intervals to pick \( N \in \mathbb{N} \) so that \( b_N - a_N < \varepsilon \). The nestedness of the intervals implies \( a_N \leq a_n < b_n \leq b_N \) for all \( n \geq N \). Therefore
\[
a_N \leq \max \{a_n : n \geq N\} = \alpha \leq b_N \text{ and } a_N \leq \min \{b_n : n \geq N\} = \beta \leq b_N.
\]
This shows \( |\alpha - \beta| \leq |b_N - a_N| < \varepsilon \). Since \( \varepsilon > 0 \) was chosen arbitrarily, we conclude \( \alpha = \beta \).

Let \( x = \alpha = \beta \). It remains to show that \( \bigcap_{n \in \mathbb{N}} I_n = \{x\} \).

First, we show that \( x = \bigcap_{n \in \mathbb{N}} I_n \). To do this, fix \( N \in \mathbb{N} \). Since \( a_n \) increases to \( x \), it’s clear that \( x \geq a_N \). Similarly, \( x \leq b_N \). Therefore \( x \in [a_N, b_N] \). Because \( N \) was chosen arbitrarily, it follows that \( x \in \bigcap_{n \in \mathbb{N}} I_n \).

Next, suppose there are \( x, y \in \bigcap_{n \in \mathbb{N}} I_n \) and let \( \varepsilon > 0 \). Choose \( N \in \mathbb{N} \) such that \( b_N - a_N < \varepsilon \). Then \( \{x, y\} \subseteq \bigcap_{n \in \mathbb{N}} I_n \subseteq [a_N, b_N] \) implies \( |x - y| < \varepsilon \). Since \( \varepsilon \) was chosen arbitrarily, we see \( x = y \). Therefore \( \bigcap_{n \in \mathbb{N}} I_n = \{x\} \).

**Example 3.16.** If \( I_n = (0, 1/n] \) for all \( n \in \mathbb{N} \), then the collection \( \{I_n : n \in \mathbb{N}\} \) is nested, but \( \bigcap_{n \in \mathbb{N}} I_n = \emptyset \). This shows the assumption that the intervals be closed in the Nested Interval Theorem is necessary.

**Example 3.17.** If \( I_n = [n, \infty) \) then the collection \( \{I_n : n \in \mathbb{N}\} \) is nested, but \( \bigcap_{n \in \mathbb{N}} I_n = \emptyset \). This shows the assumption that the lengths of the intervals be bounded is necessary. (It will be shown in Corollary 5.11 that when their lengths don’t go to 0, then the intersection is nonempty, but the uniqueness of \( x \) is lost.)
6. Cauchy Sequences

Often the biggest problem with showing that a sequence converges using the techniques we have seen so far is we must know ahead of time to what it converges. This is the “chicken and egg” problem mentioned above. An escape from this dilemma is provided by Cauchy sequences.

Definition 3.21. A sequence \( a_n \) is a Cauchy sequence if for all \( \varepsilon > 0 \) there is an \( N \in \mathbb{N} \) such that \( n, m \geq N \) implies \( |a_n - a_m| < \varepsilon \).

This definition is a bit more subtle than it might at first appear. It sort of says that all the terms of the sequence are close together from some point onward. The emphasis is on all the terms from some point onward. To stress this, first consider a negative example.

Example 3.18. Suppose \( a_n = \sum_{k=1}^{n} 1/k \) for \( n \in \mathbb{N} \). There’s a trick for showing the sequence \( a_n \) diverges. First, note that \( a_n \) is strictly increasing. For any \( n \in \mathbb{N} \), consider

\[
a_{2^n-1} = \sum_{k=1}^{2^n-1} \frac{1}{k} = \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \frac{1}{2^j + k} > \sum_{j=0}^{n-1} \sum_{k=0}^{2^j-1} \frac{1}{2^{j+1}} = \sum_{j=0}^{n-1} \frac{1}{2} = \frac{n}{2} \to \infty
\]

Hence, the subsequence \( a_{2^n-1} \) is unbounded and the sequence \( a_n \) diverges. (To see how this works, write out the first few sums of the form \( a_{2^n-1} \).

On the other hand, \( |a_{n+1} - a_n| = 1/(n+1) \to 0 \) and indeed, if \( m \) is fixed, \( |a_{n+m} - a_n| \to 0 \). This makes it seem as though the terms are getting close together, as in the definition of a Cauchy sequence. But, \( a_n \) is not a Cauchy sequence, as shown by the following theorem.

Theorem 3.22. A sequence converges iff it is a Cauchy sequence.

Proof. (⇒) Suppose \( a_n \to L \) and \( \varepsilon > 0 \). There is an \( N \in \mathbb{N} \) such that \( n \geq N \) implies \( |a_n - L| < \varepsilon/2 \). If \( m, n \geq N \), then

\[
|a_m - a_n| = |a_m - L + L - a_n| \leq |a_m - L| + |L - a_n| < \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

This shows \( a_n \) is a Cauchy sequence.

(⇐) Let \( a_n \) be a Cauchy sequence. First, we claim that \( a_n \) is bounded. To see this, let \( \varepsilon = 1 \) and choose \( N \in \mathbb{N} \) such that \( n, m \geq N \) implies \( |a_n - a_m| < 1 \). In this case, \( a_N - 1 < a_n < a_N + 1 \) for all \( n \geq N \), so \( \{a_n : n \geq N \} \) is a bounded set. The set \( \{a_n : n < N \} \), being finite, is also bounded. Since \( \{a_n : n \in \mathbb{N} \} \) is the union of these two bounded sets, it too must be bounded.

Because \( a_n \) is a bounded sequence, Theorem 3.16 implies it has a convergent subsequence \( b_n = a_{\sigma(n)} \to L \). Let \( \varepsilon > 0 \) and choose \( N \in \mathbb{N} \) so that \( n, m \geq N \) implies

\[|a_m - a_n| < \frac{\varepsilon}{2} \]
\[|a_n - a_m| < \varepsilon/2 \text{ and } |b_n - L| < \varepsilon/2. \text{ If } n \geq N, \text{ then } \sigma(n) \geq n \geq N \text{ and}
\]

\[|a_n - L| = |a_n - b_n + b_n - L| \leq |a_n - b_n| + |b_n - L| = |a_n - a_{\sigma(n)}| + |b_n - L| < \varepsilon/2 + \varepsilon/2 = \varepsilon.\]

Therefore, \(a_n \to L.\)

The fact that Cauchy sequences converge is yet another equivalent version of completeness. In fact, most advanced texts define completeness as “Cauchy sequences converge.” This is convenient in general spaces because the definition of a Cauchy sequence only needs the metric on the space and none of its other structure.

A typical example of the usefulness of Cauchy sequences is given below.

DEFINITION 3.23. A sequence \(x_n\) is contractive if there is a \(c \in (0, 1)\) such that

\[|x_{k+1} - x_k| \leq c|x_k - x_{k-1}| \text{ for all } k > 1.\]

\(c\) is called the contraction constant.

THEOREM 3.24. If a sequence is contractive, then it converges.

PROOF. Let \(x_k\) be a contractive sequence with contraction constant \(c \in (0, 1)\). We first claim that if \(n \in \mathbb{N}\), then

\[|x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2|.\] (3.5)

This is proved by induction. When \(n = 1\), the statement is

\[|x_1 - x_2| \leq c^0|x_1 - x_2| = |x_1 - x_2|,
\]

which is trivially true. Suppose that \(|x_n - x_{n+1}| \leq c^{n-1}|x_1 - x_2|\) for some \(n \in \mathbb{N}\). Then, from the definition of a contractive sequence and the induction hypothesis,

\[|x_{n+1} - x_{n+2}| \leq c|x_n - x_{n+1}| \leq c\left(c^{n-1}|x_1 - x_2|\right) = c^n|x_1 - x_2|.
\]

This shows the claim is true in the case \(n + 1\). Therefore, by induction, the claim is true for all \(n \in \mathbb{N}\).

To show \(x_n\) is a Cauchy sequence, let \(\varepsilon > 0\). Since \(c^n \to 0\), we can choose \(N \in \mathbb{N}\) so that

\[c^{N-1}\frac{|x_1 - x_2|}{(1 - c)} < \varepsilon.\] (3.6)

Let \(n > m \geq N\). Then

\[|x_n - x_m| = |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots - x_{m+1} + x_{m+1} - x_m| \leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|\]

Now, use (3.5) on each of these terms.

\[\leq c^{n-2}|x_1 - x_2| + c^{n-3}|x_1 - x_2| + \cdots + c^{m-1}|x_1 - x_2| = |x_1 - x_2|(c^{n-2} + c^{n-3} + \cdots + c^{m-1})
\]
Apply the formula for a geometric sum.

\[
= |x_1 - x_2| c^{m-1} \frac{1 - c^{n-m}}{1 - c}
\]

(3.7)

Use (3.6) to estimate the following.

\[
\leq |x_1 - x_2| \frac{c^{N-1}}{1 - c} < |x_1 - x_2| \frac{\varepsilon}{|x_1 - x_2|} = \varepsilon
\]

This shows \( x_n \) is a Cauchy sequence and must converge by Theorem 3.22.

**Example 3.19.** Let \(-1 < r < 1\) and define the sequence \(s_n = \sum_{k=0}^{n} r^k\). (You no doubt recognize this as the geometric series from your calculus course.) If \(r = 0\), the convergence of \(s_n\) is trivial. So, suppose \(r \neq 0\). In this case,

\[
\frac{|s_{n+1} - s_n|}{|s_n - s_{n-1}|} = \frac{r^{n+1}}{r^{n}} = |r| < 1
\]

and \(s_n\) is contractive. Theorem 3.24 implies \(s_n\) converges.

**Example 3.20.** Suppose \(f(x) = 2 + 1/x\), \(a_1 = 2\) and \(a_{n+1} = f(a_n)\) for \(n \in \mathbb{N}\). It is evident that \(a_n \geq 2\) for all \(n\). Some algebra gives

\[
\left| \frac{a_{n+1} - a_n}{a_{n} - a_{n-1}} \right| = \left| \frac{f(f(a_{n-1})) - f(a_{n-1})}{f(a_{n-1}) - a_{n-1}} \right| = \frac{1}{1 + 2a_{n-1}} \leq \frac{1}{5}.
\]

This shows \(a_n\) is a contractive sequence and, according to Theorem 3.24, \(a_n \to L\) for some \(L \geq 2\). Since, \(a_{n+1} = 2 + 1/a_n\), taking the limit as \(n \to \infty\) of both sides gives \(L = 2 + 1/L\). A bit more algebra shows \(L = 1 + \sqrt{2}\).

\(L\) is called a fixed point of the function \(f\); i.e. \(f(L) = L\). Many approximation techniques for solving equations involve such iterative techniques depending upon contraction to find fixed points.

The calculations in the proof of Theorem 3.24 give the means to approximate the fixed point to within an allowable error. Looking at line (3.7), notice

\[
|x_n - x_m| < |x_1 - x_2| \frac{c^{m-1}}{1 - c}.
\]

Let \(n \to \infty\) in this inequality to arrive at the error estimate

\[
|L - x_m| \leq |x_1 - x_2| \frac{c^{m-1}}{1 - c}.
\]

(3.8)

In Example 3.20, \(a_1 = 2\), \(a_2 = 5/2\) and \(c \leq 1/5\). Suppose we want to approximate \(L\) to 5 decimal places of accuracy. It suffices to find \(n\) satisfying \(|a_n - L| < 5 \times 10^{-6}\). Using (3.8), with \(m = 9\) shows

\[
|a_1 - a_2| \frac{c^{m-1}}{1 - c} \leq 1.6 \times 10^{-6}.
\]
Some arithmetic gives $a_9 \approx 2.41421$. The calculator value of

$$L = 1 + \sqrt{2} \approx 2.414213562,$$

confirming our estimate.

7. Exercises

3.1. Let the sequence $a_n = \frac{6n-1}{3n+2}$. Use the definition of convergence for a sequence to show $a_n$ converges.

3.2. If $a_n$ is a sequence such that $a_{2n} \to L$ and $a_{2n+1} \to L$, then $a_n \to L$.

3.3. Let $a_n$ be a sequence such that $a_{2n} \to A$ and $a_{2n} - a_{2n-1} \to 0$. Then $a_n \to A$.

3.4. If $a_n$ is a sequence of positive numbers converging to 0, then $\sqrt[n]{a_n} \to 0$.

3.5. Find examples of sequences $a_n$ and $b_n$ such that $a_n \to 0$ and $b_n \to \infty$ such that

(a) $a_n b_n \to 0$
(b) $a_n b_n \to \infty$
(c) $\lim_{n \to \infty} a_n b_n$ does not exist, but $a_n b_n$ is bounded.
(d) Given $c \in \mathbb{R}$, $a_n b_n \to c$.

3.6. If $x_n$ and $y_n$ are sequences such that $\lim_{n \to \infty} x_n = L \neq 0$ and $\lim_{n \to \infty} x_n y_n$ exists, then $\lim_{n \to \infty} y_n$ exists.

3.7. Determine the limit of $a_n = \sqrt[n]{n!}$. (Hint: If $n$ is even, then $n! > (n/2)^{n/2}$.)

3.8. If $\sigma : \mathbb{N} \to \mathbb{N}$ is strictly increasing, then $\sigma(n) \geq n$ for all $n \in \mathbb{N}$.

3.9. Analyze the sequence given by $a_n = \sum_{k=n+1}^{2n} 1/k$.

3.10. Every unbounded sequence contains a monotonic subsequence.

3.11. Find a sequence $a_n$ such that given $x \in [0,1]$, there is a subsequence $b_n$ of $a_n$ such that $b_n \to x$.

3.12. A sequence $a_n$ converges to 0 iff $|a_n|$ converges to 0.

3.13. Define the sequence $a_n = \sqrt{n}$ for $n \in \mathbb{N}$. Show that $|a_{n+1} - a_n| \to 0$, but $a_n$ is not a Cauchy sequence.

3.14. Suppose a sequence is defined by $a_1 = 0$, $a_1 = 1$ and $a_{n+1} = \frac{1}{2}(a_n + a_{n-1})$ for $n \geq 2$. Prove $a_n$ converges, and determine its limit.

3.15. If the sequence $a_n$ is defined recursively by $a_1 = 1$ and $a_{n+1} = \sqrt{a_n + 1}$, then show $a_n$ converges and determine its limit.
3.16. Let $a_1 = 3$ and $a_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Analyze the sequence.

3.17. If $a_n$ is a sequence such that $\lim_{n \to \infty} |a_{n+1}/a_n| = \rho < 1$, then $a_n \to 0$.

3.18. Prove that the sequence $a_n = n^3/n!$ converges.

3.19. Let $a_n$ and $b_n$ be sequences. Prove that both sequences $a_n$ and $b_n$ converge iff both $a_n + b_n$ and $a_n - b_n$ converge.

3.20. Let $a_n$ be a bounded sequence. Prove that given any $\varepsilon > 0$, there is an interval $I$ with length $\varepsilon$ such that $\{ n : a_n \in I \}$ is infinite. Is it necessary that $a_n$ be bounded?

3.21. A sequence $a_n$ converges in the mean if $\overline{a}_n = \frac{1}{n} \sum_{k=1}^{n} a_k$ converges. Prove that if $a_n \to L$, then $\overline{a}_n \to L$, but the converse is not true.

3.22. Find a sequence $x_n$ such that for all $n \in \mathbb{N}$ there is a subsequence of $x_n$ converging to $n$.

3.23. If $a_n$ is a Cauchy sequence whose terms are integers, what can you say about the sequence?

3.24. Show $a_n = \sum_{k=0}^{n} 1/k!$ is a Cauchy sequence.

3.25. If $a_n$ is a sequence such that every subsequence of $a_n$ has a further subsequence converging to $L$, then $a_n \to L$.

3.26. If $a, b \in (0, \infty)$, then show $\sqrt[n]{a^n + b^n} \to \max\{a, b\}$.

3.27. If $0 < a < 1$ and $s_n$ is a sequence satisfying $|s_{n+1}| < a|s_n|$, then $s_n \to 0$.

3.28. If $c \geq 1$ in the definition of a contractive sequence, can the sequence converge?

3.29. If $a_n$ is a convergent sequence and $b_n$ is a sequence such that $|a_m - a_n| \geq |b_m - b_n|$ for all $m, n \in \mathbb{N}$, then $b_n$ converges.

3.30. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \to L$, then $\sqrt{a_n} \to \sqrt{L}$.

3.31. If $a_n$ is a Cauchy sequence and $b_n$ is a subsequence of $a_n$ such that $b_n \to L$, then $a_n \to L$.

3.32. Let $x_1 = 3$ and $x_{n+1} = 2 - 1/x_n$ for $n \in \mathbb{N}$. Analyze the sequence.

3.33. Let $a_n$ be a sequence. $a_n \to L$ iff $\limsup a_n = L = \liminf a_n$.

3.34. Is $\limsup(a_n + b_n) = \limsup a_n + \limsup b_n$?

3.35. If $a_n$ is a sequence of positive numbers, then $\liminf a_n = \limsup 1/a_n$. 

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3.36. \( \limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n \)

3.37. \( a_n = 1/n \) is not contractive.

3.38. The equation \( x^3 - 4x + 2 = 0 \) has one real root lying between 0 and 1. Find a sequence of rational numbers converging to this root. Use this sequence to approximate the root to five decimal places.

3.39. Approximate a solution of \( x^3 - 5x + 1 = 0 \) to within \( 10^{-4} \) using a Cauchy sequence.

3.40. Prove or give a counterexample: If \( a_n \to L \) and \( \sigma : \mathbb{N} \to \mathbb{N} \) is bijective, then \( b_n = a_{\sigma(n)} \) converges. Note that \( b_n \) might not be a subsequence of \( a_n \). (\( b_n \) is called a rearrangement of \( a_n \).)