CHAPTER 4

Series

Given a sequence $a_n$, in many contexts it is natural to ask about the sum of all the numbers in the sequence. If only a finite number of the $a_n$ are nonzero, this is trivial—and not very interesting. If an infinite number of the terms aren’t zero, the path becomes less obvious. Indeed, it’s even somewhat questionable whether it makes sense at all to add an infinite number of numbers.

There are many approaches to this question. The method given below is the most common technique. Others are mentioned in the exercises.

1. What is a Series?

The idea behind adding up an infinite collection of numbers is a reduction to the well-understood idea of a sequence. This is a typical approach in mathematics: reduce a question to a previously solved problem.

**Definition 4.1.** Given a sequence $a_n$, the series having $a_n$ as its terms is the new sequence

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

The numbers $s_n$ are called the partial sums of the series. If $s_n \to S \in \mathbb{R}$, then the series converges to $S$. This is normally written as

$$\sum_{k=1}^{\infty} a_k = S.$$

Otherwise, the series diverges.

The notation $\sum_{n=1}^{\infty} a_n$ is understood to stand for the sequence of partial sums of the series with terms $a_n$. When there is no ambiguity, this is often abbreviated to just $\sum a_n$.

**Example 4.1.** If $a_n = (-1)^n$ for $n \in \mathbb{N}$, then $s_1 = -1$, $s_2 = -1 + 1 = 0$, $s_3 = -1 + 1 - 1 = -1$ and in general

$$s_n = \frac{(-1)^n - 1}{2}$$

does not converge because it oscillates between $-1$ and $0$. Therefore, the series $\sum (-1)^n$ diverges.
Example 4.2 (Geometric Series). Recall that a sequence of the form $a_n = cr^{n-1}$ is called a geometric sequence. It gives rise to a series
\[ \sum_{n=1}^{\infty} cr^{n-1} = c + cr + cr^2 + cr^3 + \cdots \]
called a geometric series. The number $r$ is called the ratio of the series.

Suppose $a_n = r^{n-1}$ for $r \neq 1$. Then,
\[ s_1 = 1, \quad s_2 = 1 + r, \quad s_3 = 1 + r + r^2, \quad \ldots \]
In general, it can be shown by induction (or even long division of polynomials) that
\[ s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} r^{k-1} = \frac{1 - r^n}{1 - r}. \tag{4.1} \]
The convergence of $s_n$ in (4.1) depends on the value of $r$. Letting $n \to \infty$, it’s apparent that $s_n$ diverges when $|r| > 1$ and converges to $1/(1 - r)$ when $|r| < 1$.

When $r = 1$, $s_n = n \to \infty$. When $r = -1$, it’s essentially the same as Example 4.1, and therefore diverges. In summary,
\[ \sum_{n=1}^{\infty} cr^{n-1} = \frac{c}{1 - r} \]
for $|r| < 1$, and diverges when $|r| \geq 1$. This is called a geometric series with ratio $r$.

**Figure 4.1.** Stepping to the wall.

In some cases, the geometric series has an intuitively plausible limit. If you start two meters away from a wall and keep stepping halfway to the wall, no number of steps will get you to the wall, but a large number of steps will get you as close to the wall as you want. (See Figure 4.1.) So, the total distance stepped has limiting value 2. The total distance after $n$ steps is the $n$th partial sum of a geometric series with ratio $r = 1/2$ and $c = 1$.

Example 4.3 (Harmonic Series). The series $\sum_{n=1}^{\infty} 1/n$ is called the harmonic series. It was shown in Example 3.18 that the harmonic series diverges.

Example 4.4. The terms of the sequence
\[ a_n = \frac{1}{n^2 + n}, \quad n \in \mathbb{N}. \]
can be decomposed into partial fractions as

\[ a_n = \frac{1}{n} - \frac{1}{n+1}. \]

If \( s_n \) is the series having \( a_n \) as its terms, then \( s_1 = 1/2 = 1 - 1/2 \). We claim that \( s_n = 1 - 1/(n+1) \) for all \( n \in \mathbb{N} \). To see this, suppose \( s_k = 1 - 1/(k+1) \) for some \( k \in \mathbb{N} \). Then

\[ s_{k+1} = s_k + a_{k+1} = 1 - \frac{1}{k+1} + \left( \frac{1}{k+1} - \frac{1}{k+2} \right) = 1 - \frac{1}{k+2} \]

and the claim is established by induction. Now it’s easy to see that

\[ \sum_{n=1}^{\infty} \frac{1}{n^2 + n} = \lim_{n \to \infty} \left( 1 - \frac{1}{n+2} \right) = 1. \]

This is an example of a *telescoping* series. The name is apparently based on the idea that the middle terms of the series cancel, causing the series to collapse like a hand-held telescope.

The following theorem is an easy consequence of the properties of sequences shown in Theorem 3.8.

**Theorem 4.2.** Let \( \sum a_n \) and \( \sum b_n \) be convergent series.

(a) If \( c \in \mathbb{R} \), then \( \sum c a_n = c \sum a_n \).

(b) \( \sum (a_n + b_n) = \sum a_n + \sum b_n \).

(c) \( a_n \to 0 \)

**Proof.** Let \( A_n = \sum_{k=1}^{n} a_k \) and \( B_n = \sum_{k=1}^{n} b_k \) be the sequences of partial sums for each of the two series. By assumption, there are numbers \( A \) and \( B \) where \( A_n \to A \) and \( B_n \to B \).

(a) \( \sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k = c A_n \to c A \).

(b) \( \sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k = A_n + B_n \to A + B \).

(c) For \( n > 1 \), \( a_n = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n-1} a_k = A_n - A_{n-1} \to A - A = 0 \).

Notice that the first two parts of Theorem 4.2 show that the set of all convergent series is closed under linear combinations.

Theorem 4.2(c) is very useful because its contrapositive provides the most basic test for divergence.

**Corollary 4.3 (Going to Zero Test).** If \( a_n \not\to 0 \), then \( \sum a_n \) diverges.

Many have made the mistake of reading too much into Corollary 4.3. It can only be used to show divergence. When the terms of a series do tend to zero, that does not guarantee convergence. Example 4.3, shows Theorem 4.2(c) is necessary, but not sufficient for convergence.

Another useful observation is that the partial sums of a convergent series are a Cauchy sequence. The Cauchy criterion for sequences can be rephrased for series as the following theorem, the proof of which is Exercise 4.4.
**THEOREM 4.4** (Cauchy Criterion for Series). Let $\sum a_n$ be a series. The following statements are equivalent.

(a) $\sum a_n$ converges.

(b) For every $\varepsilon > 0$ there is an $N \in \mathbb{N}$ such that whenever $n \geq m \geq N$, then

$$\left| \sum_{i=m}^{n} a_i \right| < \varepsilon.$$

### 2. Positive Series

Most of the time, it is very hard or impossible to determine the exact limit of a convergent series. We must satisfy ourselves with determining whether a series converges, and then approximating its sum. For this reason, the study of series usually involves learning a collection of theorems that might answer whether a given series converges, but don’t tell us to what it converges. These theorems are usually called the *convergence tests*. The reader probably remembers a battery of such tests from her calculus course. There is a myriad of such tests, and the standard ones are presented in the next few sections, along with a few of those less widely used.

Since convergence of a series is determined by convergence of the sequence of its partial sums, the easiest series to study are those with well-behaved partial sums. Series with monotone sequences of partial sums are certainly the simplest such series.

**DEFINITION 4.5.** The series $\sum a_n$ is a *positive series* if $a_n \geq 0$ for all $n$.

The advantage of a positive series is that its sequence of partial sums is nonnegative and increasing. Since an increasing sequence converges if and only if it is bounded above, there is a simple criterion to determine whether a positive series converges. All of the standard convergence tests for positive series exploit this criterion.

#### 2.1. The Most Common Convergence Tests.

All beginning calculus courses contain several simple tests to determine whether positive series converge. Most of them are presented below.

2.1.1. *Comparison Tests.* The most basic convergence tests are the comparison tests. In these tests, the behavior of one series is inferred from that of another series. Although they’re easy to use, there is one often fatal catch: in order to use a comparison test, you must have a known series to which you can compare the mystery series. For this reason, a wise mathematician collects example series for her toolbox. The more samples in the toolbox, the more powerful are the comparison tests.

**THEOREM 4.6** (Comparison Test). Suppose $\sum a_n$ and $\sum b_n$ are positive series with $a_n \leq b_n$ for all $n$.

(a) If $\sum b_n$ converges, then so does $\sum a_n$.

(b) If $\sum a_n$ diverges, then so does $\sum b_n$. 

Positive Series

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots + a_{15} + a_{16} + \cdots \leq 2a_2 \]

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots + a_{15} + a_{16} + \cdots \geq a_2 \]

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots + a_{15} + a_{16} + \cdots \leq 4a_4 \]

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots + a_{15} + a_{16} + \cdots \geq 4a_8 \]

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots + a_{15} + a_{16} + \cdots \leq 8a_8 \]

\[ a_1 + a_2 + a_3 + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 + \cdots + a_{15} + a_{16} + \cdots \geq 8a_{16} \]

**Figure 4.2.** This diagram shows the groupings used in inequality (4.3).

**Proof.** Let \( A_n \) and \( B_n \) be the partial sums of \( \sum a_n \) and \( \sum b_n \), respectively. It follows from the assumptions that \( A_n \) and \( B_n \) are increasing and for all \( n \in \mathbb{N} \),

(4.2) \[ A_n \leq B_n. \]

If \( \sum b_n = B \), then (4.2) implies \( B \) is an upper bound for \( A_n \), and \( \sum a_n \) converges.

On the other hand, if \( \sum a_n \) diverges, \( A_n \to \infty \) and the Sandwich Theorem 3.9(b) shows \( B_n \to \infty \).

**Example 4.5.** Example 4.3 shows that \( \sum 1/n^p \) diverges. If \( p \leq 1 \), then \( 1/n^p \geq 1/n \), and Theorem 4.6 implies \( \sum 1/n^p \) diverges.

**Example 4.6.** The series \( \sum \sin^2 n/2^n \) converges because

\[ \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n} \]

for all \( n \) and the geometric series \( \sum 1/2^n = 1 \).

**Theorem 4.7 (Cauchy’s Condensation Test\(^1\)).** Suppose \( a_n \) is a decreasing sequence of nonnegative numbers. Then

\[ \sum a_n \text{ converges iff } \sum 2^n a_{2^n} \text{ converges.} \]

**Proof.** Since \( a_n \) is decreasing, for \( n \in \mathbb{N} \),

(4.3) \[ \sum_{k=2^n}^{2^{n+1}-1} a_k \leq 2^n a_{2^n} \leq 2 \sum_{k=2^n}^{2^{n+1}-1} a_k. \]

(See Figure 2.1.1.) Adding for \( 1 \leq n \leq m \) gives

(4.4) \[ \sum_{k=2}^{2^{m+1}-1} a_k \leq \sum_{k=1}^{m} 2^k a_{2^k} \leq 2 \sum_{k=1}^{2m-1} a_k. \]

Suppose \( \sum a_n \) converges to \( S \). The right-hand inequality of (4.4) shows \( \sum_{k=1}^{m} 2^k a_{2^k} < 2S \) and \( \sum 2^k a_{2^k} \) must converge. On the other hand, if \( \sum a_n \) diverges, then the left-hand side of (4.4) is unbounded, forcing \( \sum 2^k a_{2^k} \) to diverge.

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\(^1\)The series \( \sum 2^n a_{2^n} \) is sometimes called the condensed series associated with \( \sum a_n \).

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http://math.louisville.edu/~lee/ira
Example 4.7 (p-series). For fixed $p \in \mathbb{R}$, the series $\sum 1/n^p$ is called a \textit{p-series}. The special case when $p = 1$ is the \textit{harmonic} series. Notice

$$\sum \frac{2^n}{(2^n)^p} = \sum (2^{1-p})^n$$

is a geometric series with ratio $2^{1-p}$, so it converges only when $2^{1-p} < 1$. Since $2^{1-p} < 1$ only when $p > 1$, it follows from the Cauchy Condensation Test that the \textit{p-series} converges when $p > 1$ and diverges when $p \leq 1$. (Of course, the divergence half of this was already known from Example 4.5.)

The \textit{p-series} are often useful for the Comparison Test, and also occur in many areas of advanced mathematics such as harmonic analysis and number theory.

Theorem 4.8 (Limit Comparison Test). Suppose $\sum a_n$ and $\sum b_n$ are positive series with

$$a = \liminf \frac{a_n}{b_n} \leq \limsup \frac{a_n}{b_n} = \beta.$$ 

(a) If $\alpha \in (0, \infty)$ and $\sum a_n$ converges, then so does $\sum b_n$, and if $\sum b_n$ diverges, then so does $\sum a_n$.

(b) If $\beta \in (0, \infty)$ and $\sum b_n$ diverges, then so does $\sum a_n$, and if $\sum a_n$ converges, then so does $\sum b_n$.

Proof. To prove (a), suppose $\alpha > 0$. There is an $N \in \mathbb{N}$ such that

$$n \geq N \implies \frac{\alpha}{2} < \frac{a_n}{b_n}.$$ 

If $n > N$, then (4.6) gives

$$\frac{\alpha}{2} \sum_{k=N}^{n} b_k < \sum_{k=N}^{n} a_k.$$ 

If $\sum a_n$ converges, then (4.7) shows the partial sums of $\sum b_n$ are bounded and $\sum b_n$ converges. If $\sum b_n$ diverges, then (4.7) shows the partial sums of $\sum a_n$ are unbounded, and $\sum a_n$ must diverge.

The proof of (b) is similar. \hfill \Box

The following easy corollary is the form this test takes in most calculus books. It’s easier to use than Theorem 4.8 and suffices most of the time.

Corollary 4.9 (Limit Comparison Test). Suppose $\sum a_n$ and $\sum b_n$ are positive series with

$$a = \lim_{n \to \infty} \frac{a_n}{b_n}.$$ 

If $\alpha \in (0, \infty)$, then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Example 4.8. To test the series $\sum \frac{1}{2^n - n}$ for convergence, let

$$a_n = \frac{1}{2^n - n} \text{ and } b_n = \frac{1}{2^n}.$$
Then
\[
\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{1/(2^n - n)}{1/2^n} = \lim_{n \to \infty} \frac{2^n}{2^n - n} = \lim_{n \to \infty} \frac{1}{1 - n/2^n} = 1 \in (0, \infty).
\]
Since \(1/2^n = 1\), the original series converges by the Limit Comparison Test.

2.1.2. Geometric Series-Type Tests. The most important series is undoubtedly the geometric series. Several standard tests are basically comparisons to geometric series.

**Theorem 4.10 (Root Test).** Suppose \(\sum a_n\) is a positive series and 
\[
\rho = \limsup_{n} a_n^{1/n}.
\]
If \(\rho < 1\), then \(\sum a_n\) converges. If \(\rho > 1\), then \(\sum a_n\) diverges.

**Proof.** First, suppose \(\rho < 1\) and \(r \in (\rho, 1)\). There exists an \(N \in \mathbb{N}\) so that \(a_n^{1/n} < r\) for all \(n \geq N\). This is the same as \(a_n < r^n\) for all \(n \geq N\). Using this, it follows that when \(n \geq N\),
\[
\sum_{k=1}^{n} a_k = \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} a_k < \sum_{k=1}^{N-1} a_k + \sum_{k=N}^{n} r^k < \sum_{k=1}^{N-1} a_k + \frac{r^N}{1 - r}.
\]
This shows the partial sums of \(\sum a_n\) are bounded. Therefore, it must converge.

If \(\rho > 1\), there is an increasing sequence of integers \(k_n \to \infty\) such that \(a_n^{1/k_n} > 1\) for all \(n \in \mathbb{N}\). This shows \(a_{k_n} > 1\) for all \(n \in \mathbb{N}\). By Theorem 4.3, \(\sum a_n\) diverges. 

**Example 4.9.** For any \(x \in \mathbb{R}\), the series \(\sum |x^n|/n!\) converges. To see this, note that according to Exercise 3.3.7,
\[
\left(\frac{|x^n|}{n!}\right)^{1/n} = \frac{|x|}{(n!)^{1/n}} \to 0 < 1.
\]
Applying the Root Test shows the series converges.

**Example 4.10.** Consider the \(p\)-series \(\sum 1/n\) and \(\sum 1/n^2\). The first diverges and the second converges. Since \(n^{1/n} \to 1\) and \(n^{2/n} \to 1\), it can be seen that when \(\rho = 1\), the Root Test is inconclusive.

**Theorem 4.11 (Ratio Test).** Suppose \(\sum a_n\) is a positive series. Let
\[
r = \liminf_{n} \frac{a_{n+1}}{a_n} \leq \limsup_{n} \frac{a_{n+1}}{a_n} = R.
\]
If \(R < 1\), then \(\sum a_n\) converges. If \(r > 1\), then \(\sum a_n\) diverges.

**Proof.** First, suppose \(R < 1\) and \(r \in (R, 1)\). There exists \(N \in \mathbb{N}\) such that \(a_{n+1}/a_n < \rho\) whenever \(n \geq N\). This implies \(a_{n+1} < \rho\, a_n\) whenever \(n \geq N\). From this it’s easy to prove by induction that \(a_{N+m} < \rho^m a_N\) whenever \(m \in \mathbb{N}\). It follows
that, for \( n > N \),
\[
\sum_{k=1}^{n} a_k = \sum_{k=1}^{N} a_k + \sum_{k=N+1}^{n} a_k \\
= \sum_{k=1}^{N} a_k + \sum_{k=1}^{n-N} a_{N+k} \\
< \sum_{k=1}^{N} a_k + \sum_{k=1}^{n-N} a_N \rho^k \\
< \sum_{k=1}^{N} a_k + \frac{a_N \rho}{1 - \rho}.
\]

Therefore, the partial sums of \( \sum a_n \) are bounded, and \( \sum a_n \) converges.

If \( r > 1 \), then choose \( N \in \mathbb{N} \) so that \( a_{n+1} > a_n \) for all \( n \geq N \). It’s now apparent that \( a_n \neq 0 \). \( \square \)

In calculus books, the ratio test usually takes the following simpler form.

**Corollary 4.12 (Ratio Test).** Suppose \( \sum a_n \) is a positive series. Let
\[
r = \lim_{n \to \infty} \frac{a_{n+1}}{a_n}.
\]

If \( r < 1 \), then \( \sum a_n \) converges. If \( r > 1 \), then \( \sum a_n \) diverges.

From a practical viewpoint, the ratio test is often easier to apply than the root test. But, the root test is actually the stronger of the two in the sense that there are series for which the ratio test fails, but the root test succeeds. (See Exercise 4.10, for example.) This happens because

\[
\liminf_a_n \leq \limsup a_n^1/n \leq \limsup a_n \leq \limsup a_{n+1}/a_n.
\]

To see this, note the middle inequality is always true. To prove the right-hand inequality, choose \( r = \lim \sup a_{n+1}/a_n \). It suffices to show \( \lim \sup a_n^{1/n} \leq r \). As in the proof of the ratio test, \( a_{n+k} < r^k a_n \). This implies
\[
a_{n+k} < r^{n+k} \frac{a_n}{r^n},
\]
which leads to
\[
a_n^{1/(n+k)} < r \left( \frac{a_n}{r^n} \right)^{1/(n+k)}.
\]

Finally,
\[
\limsup a_n^{1/n} = \limsup_{k \to \infty} a_n^{1/(n+k)} \leq \limsup_{k \to \infty} r \left( \frac{a_n}{r^n} \right)^{1/(n+k)} = r.
\]

The left-hand inequality is proved similarly.
2.2. Kummer-Type Tests.

This is an advanced section that can be omitted.

Most times the simple tests of the preceding section suffice. However, more difficult series require more delicate tests. There dozens of other, more specialized, convergence tests. Several of them are consequences of the following theorem.

**Theorem 4.13 (Kummer’s Test).** Suppose \( \sum a_n \) is a positive series, \( p_n \) is a sequence of positive numbers and

\[
\alpha = \liminf \left( p_n \frac{a_n}{a_{n+1}} - p_{n+1} \right) \leq \limsup \left( p_n \frac{a_n}{a_{n+1}} - p_{n+1} \right) = \beta
\]

If \( \alpha > 0 \), then \( \sum a_n \) converges. If \( \sum 1/p_n \) diverges and \( \beta < 0 \), then \( \sum a_n \) diverges.

**Proof.** Let \( s_n = \sum_{k=1}^{n} a_k \), suppose \( \alpha > 0 \) and choose \( r \in (0, \alpha) \). There must be an \( N > 1 \) such that

\[
p_n \frac{a_n}{a_{n+1}} - p_{n+1} > r, \forall n \geq N.
\]

Rearranging this gives

\[
p_n a_n - p_{n+1} a_{n+1} > r a_{n+1}, \forall n \geq N.
\]

For \( M > N \), (4.11) implies

\[
\sum_{n=N}^{M} (p_n a_n - p_{n+1} a_{n+1}) > \sum_{n=N}^{M} r a_{n+1}
\]

\[
p_N a_N - p_{M+1} a_{M+1} > r (s_M - s_{N-1})
\]

\[
p_N a_N - p_{M+1} a_{M+1} + r s_{N-1} > r s_M
\]

\[
\frac{p_N a_N + r s_{N-1}}{r} > s_M
\]

Since \( N \) is fixed, the left side is an upper bound for \( s_M \), and it follows that \( \sum a_n \) converges.

Next suppose \( \sum 1/p_n \) diverges and \( \beta < 0 \). There must be an \( N \in \mathbb{N} \) such that

\[
p_n \frac{a_n}{a_{n+1}} - p_{n+1} < 0, \forall n \geq N.
\]

This implies

\[
p_n a_n < p_{n+1} a_{n+1}, \forall n \geq N.
\]

Therefore, \( p_n a_n > p_N a_N \) whenever \( n > N \) and

\[
a_n > p_N a_N \frac{1}{p_n}, \forall n \geq N.
\]

Because \( N \) is fixed and \( \sum 1/p_n \) diverges, the Comparison Test shows \( \sum a_n \) diverges. \( \square \)
Kummer’s test is powerful. In fact, it can be shown that, given any positive series, a judicious choice of the sequence \(p_n\) can always be made to determine whether it converges. (See Exercise 4.17, [20] and [19].) But, as stated, Kummer’s test is not very useful because choosing \(p_n\) for a given series is often difficult. Experience has led to some standard choices that work with large classes of series. For example, Exercise 4.9 asks you to prove the choice \(p_n = 1\) for all \(n\) reduces Kummer’s test to the standard ratio test. Other useful choices are shown in the following theorems.

**Theorem 4.14 (Raabe’s Test).** Let \(\sum a_n\) be a positive series such that \(a_n > 0\) for all \(n\). Define

\[
\alpha = \limsup_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) \geq \liminf_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) = \beta
\]

If \(\alpha > 1\), then \(\sum a_n\) converges. If \(\beta < 1\), then \(\sum a_n\) diverges.

**Proof.** Let \(p_n = n\) in Kummer’s test, Theorem 4.13. \(\square\)

When Raabe’s test is inconclusive, there are even more delicate tests, such as the theorem given below.

**Theorem 4.15 (Bertrand’s Test).** Let \(\sum a_n\) be a positive series such that \(a_n > 0\) for all \(n\). Define

\[
\alpha = \liminf_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) \leq \limsup_{n \to \infty} n \left( \frac{a_n}{a_{n+1}} - 1 \right) = \beta
\]

If \(\alpha > 1\), then \(\sum a_n\) converges. If \(\beta < 1\), then \(\sum a_n\) diverges.

**Proof.** Let \(p_n = n \ln n\) in Kummer’s test. \(\square\)

**Example 4.11.** Consider the series

\[
\sum a_n = \sum \left( \prod_{k=1}^{n} \frac{2k}{2k+1} \right)^p.
\]

It’s of interest to know for what values of \(p\) it converges.

An easy computation shows that \(a_{n+1}/a_n \to 1\), so the ratio test is inconclusive. Next, try Raabe’s test. Manipulating

\[
\lim_{n \to \infty} n \left( \left( \frac{a_n}{a_{n+1}} \right)^p - 1 \right) = \lim_{n \to \infty} n \left( \frac{2n+3}{2n+2} \right)^p - 1
\]

it becomes a \(0/0\) form and can be evaluated with L’Hospital’s rule.

\[
\lim_{n \to \infty} n^2 \left( \frac{3+2n}{2+2n} \right)^p = \frac{p}{2}.
\]

From Raabe’s test, Theorem 4.14, it follows that the series converges when \(p > 2\) and diverges when \(p < 2\). Raabe’s test is inconclusive when \(p = 2\).

\[\text{2See §5.2.}\]
Now, suppose $p = 2$. Consider
\[
\lim_{n \to \infty} \ln \left( n \left( \frac{a_n}{a_{n+1}} - 1 \right) - 1 \right) = - \lim_{n \to \infty} \ln \frac{(4 + 3n)}{4(n+1)^2} = 0
\]
and Bertrand’s test, Theorem 4.15, shows divergence.

The series (4.12) converges only when $p > 2$.

3. Absolute and Conditional Convergence

The tests given above are for the restricted case when a series has positive terms. If the stipulation that the series be positive is thrown out, things becomes considerably more complicated. But, as is often the case in mathematics, some problems can be attacked by reducing them to previously solved cases. The following definition and theorem show how to do this for some special cases.

**Definition 4.16.** Let $\sum a_n$ be a series. If $\sum |a_n|$ converges, then $\sum a_n$ is **absolutely convergent**. If it is convergent, but not absolutely convergent, then it is **conditionally convergent**.

Since $\sum |a_n|$ is a positive series, the preceding tests can be used to determine its convergence. The following theorem shows that this is also enough for convergence of the original series.

**Theorem 4.17.** If $\sum a_n$ is absolutely convergent, then it is convergent.

**Proof.** Let $\varepsilon > 0$. Theorem 4.4 yields an $N \in \mathbb{N}$ such that when $n \geq m \geq N$,
\[
\varepsilon > \sum_{k=m}^{n} |a_k| \geq \sum_{k=m}^{n} a_k \geq 0.
\]
Another application Theorem 4.4 finishes the proof. \(\square\)

**Example 4.12.** The series $\sum (-1)^{n+1}/n$ is called the **alternating harmonic series**. (See Figure 4.3.) Since the harmonic series diverges, we see the alternating harmonic series is not absolutely convergent.

On the other hand, if $s_n = \sum_{k=1}^{n} (-1)^{k+1}/k$, then
\[
s_{2n} = \sum_{k=1}^{n} \left( \frac{1}{2k-1} - \frac{1}{2k} \right) = \sum_{k=1}^{n} \frac{1}{2k(2k-1)}
\]
is a positive series that converges by the Comparison Test. Since $|s_{2n} - s_{2n-1}| = 1/2n \to 0$, it’s clear that $s_{2n-1}$ must also converge to the same limit. Therefore, $s_n$ converges and $\sum (-1)^{n+1}/n$ is conditionally convergent. (Another way to show the alternating harmonic series converges is shown in Example 3.18.)

To summarize: absolute convergence implies convergence, but convergence does not imply absolute convergence.

There are a few tests that address conditional convergence. Following are the most well-known.

**Theorem 4.18 (Abel’s Test).** Let $a_n$ and $b_n$ be sequences satisfying
This plot shows the first 35 partial sums of the alternating harmonic series. It can be shown it converges to $\ln 2 \approx 0.6931$, which is the level of the dashed line. Notice how the odd partial sums decrease to $\ln 2$ and the even partial sums increase to $\ln 2$.

(a) $s_n = \sum_{k=1}^{n} a_k$ is a bounded sequence.
(b) $b_n \geq b_{n+1}$, $\forall n \in \mathbb{N}$
(c) $b_n \to 0$

Then $\sum a_n b_n$ converges.

To prove this theorem, the following lemma is needed.

**Lemma 4.19 (Summation by Parts).** *For every pair of sequences $a_n$ and $b_n$,*

$$\sum_{k=1}^{n} a_k b_k = b_{n+1} \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} (b_{k+1} - b_k) \sum_{\ell=1}^{k} a_{\ell}$$

**Proof.** Let $s_0 = 0$ and $s_n = \sum_{k=1}^{n} a_k$ when $n \in \mathbb{N}$. Then

$$\sum_{k=1}^{n} a_k b_k = \sum_{k=1}^{n} (s_k - s_{k-1}) b_k$$

$$= \sum_{k=1}^{n} s_k b_k - \sum_{k=1}^{n} s_{k-1} b_k$$

$$= \sum_{k=1}^{n} s_k b_k - \left( \sum_{k=1}^{n} s_k b_{k+1} - s_n b_{n+1} \right)$$

$$= b_{n+1} \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} (b_{k+1} - b_k) \sum_{\ell=1}^{k} a_{\ell}$$
PROOF. To prove the theorem, suppose \( \sum_{k=1}^{n} a_k < M \) for all \( n \in \mathbb{N} \). Let \( \varepsilon > 0 \) and choose \( N \in \mathbb{N} \) such that \( b_N < \varepsilon/2M \). If \( N \leq m < n \), use Lemma 4.19 to write
\[
\left| \sum_{\ell=m}^{n} a_\ell b_\ell \right| = \left| \sum_{\ell=1}^{n} a_\ell b_\ell - \sum_{\ell=1}^{m-1} a_\ell b_\ell \right|
\]
\[
= \left| b_{n+1} \sum_{\ell=1}^{n} a_\ell - \sum_{\ell=1}^{n} (b_{\ell+1} - b_\ell) \sum_{k=1}^{\ell} a_k \right|
\]
\[
- \left( b_m \sum_{\ell=1}^{m-1} a_\ell - \sum_{\ell=1}^{m-1} (b_{\ell+1} - b_\ell) \sum_{k=1}^{\ell} a_k \right) \]
Using (a) gives
\[
\leq (b_{n+1} + b_m)M + M \sum_{\ell=m}^{n} |b_{\ell+1} - b_\ell| \]
Now, use (b) to see
\[
= (b_{n+1} + b_m)M + M \sum_{\ell=m}^{n} (b_\ell - b_{\ell+1}) \]
and then telescope the sum to arrive at
\[
= (b_{n+1} + b_m)M + M(b_m - b_{n+1}) \]
\[
= 2Mb_m \]
\[
< 2M \frac{\varepsilon}{2M} \]
\[
< \varepsilon \]
This shows \( \sum_{\ell=1}^{n} a_\ell b_\ell \) satisfies Theorem 4.4, and therefore converges. \( \square \)

There’s one special case of this theorem that’s most often seen in calculus texts.

COROLLARY 4.20 (Alternating Series Test). If \( c_n \) decreases to 0, then the series \( \sum (-1)^{n+1} c_n \) converges. Moreover, if \( s_n = \sum_{k=1}^{n} (-1)^{k+1} c_k \) and \( s_n \rightarrow s \), then \( |s_n - s| < c_{n+1} \).

PROOF. Let \( a_n = (-1)^{n+1} \) and \( b_n = c_n \) in Theorem 4.18 to see the series converges to some number \( s \). For \( n \in \mathbb{N} \), let \( s_n = \sum_{k=0}^{n} (-1)^{k+1} c_k \) and \( s_0 = 0 \). Since
\[
s_{2n} - s_{2n+2} = -c_{2n+1} + c_{2n+2} \leq 0 \text{ and } s_{2n+1} - s_{2n+3} = c_{2n+2} - c_{2n+3} \geq 0, \]
It must be that \( s_{2n} \uparrow s \) and \( s_{2n+1} \downarrow s \). For all \( n \in \omega \),
\[
0 \leq s_{2n+1} - s \leq s_{2n+1} - s_{2n+2} = c_{2n+2} \text{ and } 0 \leq s - s_{2n} \leq s_{2n+1} - s_{2n} = c_{2n+1}. \]
This shows \( |s_n - s| < c_{n+1} \) for all \( n \). \( \square \)
Here is a more whimsical way to visualize the partial sums of the alternating harmonic series.

A series such as that in Corollary 4.20 is called an alternating series. More formally, if \( a_n \) is a sequence such that \( a_n/a_{n+1} < 0 \) for all \( n \), then \( \sum a_n \) is an alternating series. Informally, it just means the series alternates between positive and negative terms.

Example 4.13. Corollary 4.20 provides another way to prove the alternating harmonic series in Example 4.12 converges. Figures 4.3 and 4.4 show how the partial sums bounce up and down across the sum of the series.

4. Rearrangements of Series

This is an advanced section that can be omitted.

We want to use our standard intuition about adding lists of numbers when working with series. But, this intuition has been formed by working with finite sums and does not always work with series.

Example 4.14. Suppose \( \sum (-1)^{n+1}/n = \gamma \) so that \( \sum (-1)^{n+1}2/n = 2\gamma \). It's easy to show \( \gamma > 1/2 \). Consider the following calculation.

\[
2\gamma = \sum (-1)^{n+1} \frac{2}{n} = 2 - 1 + \frac{2}{3} - \frac{1}{2} + \frac{2}{5} - \frac{1}{3} + \cdots
\]
Rearrangements of Series

Rearrange and regroup.

\[
= (2 - 1) - \frac{1}{2} + \left( \frac{2}{3} - \frac{1}{3} \right) - \frac{1}{4} + \left( \frac{2}{5} - \frac{1}{5} \right) - \frac{1}{6} + \ldots
\]

\[
= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots
\]

\[
= \gamma
\]

So, \( \gamma = 2\gamma \) with \( \gamma \neq 0 \). Obviously, rearranging and regrouping of this series is a questionable thing to do.

In order to carefully consider the problem of rearranging a series, a precise definition is needed.

**Definition 4.21.** Let \( \sigma : \mathbb{N} \rightarrow \mathbb{N} \) be a bijection and \( \sum a_n \) be a series. The new series \( \sum a_{\sigma(n)} \) is a rearrangement of the original series.

The problem with Example 4.14 is that the series is conditionally convergent. Such examples cannot happen with absolutely convergent series. For the most part, absolutely convergent series behave as we are intuitively led to expect.

**Theorem 4.22.** If \( \sum a_n \) is absolutely convergent and \( \sum a_{\sigma(n)} \) is a rearrangement of \( \sum a_n \), then \( \sum a_{\sigma(n)} = \sum a_n \).

**Proof.** Let \( \sum a_n = s \) and \( \epsilon > 0 \). Choose \( N \in \mathbb{N} \) so that \( N \leq m < n \) implies \( \sum_{k=m}^{n} |a_k| < \epsilon \). Choose \( M \geq N \) such that

\[
\{1, 2, \ldots, N\} \subset \{\sigma(1), \sigma(2), \ldots, \sigma(M)\}.
\]

If \( P > M \), then

\[
\left| \sum_{k=1}^{P} a_k - \sum_{k=1}^{P} a_{\sigma(k)} \right| \leq \sum_{k=N+1}^{\infty} |a_k| \leq \epsilon
\]

and both series converge to the same number. \( \Box \)

When a series is conditionally convergent, the result of a rearrangement is hard to predict. This is shown by the following surprising theorem.

**Theorem 4.23 (Riemann Rearrangement).** If \( \sum a_n \) is conditionally convergent and \( c \in \mathbb{R} \cup \{-\infty, \infty\} \), then there is a rearrangement \( \sigma \) such that \( \sum a_{\sigma(n)} = c \).

To prove this, the following lemma is needed.

**Lemma 4.24.** If \( \sum a_n \) is conditionally convergent and

\[
b_n = \begin{cases} a_n, & a_n > 0 \\ 0, & a_n \leq 0 \end{cases} \quad \text{and} \quad c_n = \begin{cases} -a_n, & a_n < 0 \\ 0, & a_n \geq 0 \end{cases},
\]

then both \( \sum b_n \) and \( \sum c_n \) diverge.

**Proof.** Suppose \( \sum b_n \) converges. By assumption, \( \sum a_n \) converges, so Theorem 4.2 implies

\[
\sum c_n = \sum b_n - \sum a_n
\]
converges. Another application of Theorem 4.2 shows
\[ \sum |a_n| = \sum b_n + \sum c_n \]
converges. This is a contradiction of the assumption that \( \sum a_n \) is conditionally convergent, so \( \sum b_n \) cannot converge.

A similar contradiction arises under the assumption that \( \sum c_n \) converges. □

**Proof.** (Theorem 4.23) Let \( b_n \) and \( c_n \) be as in Lemma 4.24 and define the subsequence \( a^+_n \) of \( b_n \) by removing those terms for which \( b_n = 0 \) and \( a_n \neq 0 \). Define the subsequence \( a^-_n \) of \( c_n \) by removing those terms for which \( c_n = 0 \). The series \( \sum_{n=1}^{\infty} a^+_n \) and \( \sum_{n=1}^{\infty} a^-_n \) are still divergent because only terms equal to zero have been removed from \( b_n \) and \( c_n \).

Now, let \( c \in \mathbb{R} \) and \( m_0 = n_0 = 0 \). According to Lemma 4.24, we can define the natural numbers
\[
m_1 = \min\{n : \sum_{k=1}^{n} a^+_k > c\} \text{ and } n_1 = \min\{n : \sum_{\ell=1}^{n} a^-_\ell < c\}.
\]

If \( m_p \) and \( n_p \) have been chosen for some \( p \in \mathbb{N} \), then define
\[
m_{p+1} = \min\left\{n : \sum_{k=0}^{p} \left( \sum_{\ell=m_k+1}^{m_{k+1}} a^+_\ell - \sum_{\ell=n_k+1}^{n_{k+1}} a^-_\ell \right) + \sum_{\ell=m_p+1}^{n_p+1} a^+_\ell > c\right\}
\]
and
\[
n_{p+1} = \min\left\{n : \sum_{k=0}^{p} \left( \sum_{\ell=m_k+1}^{m_{k+1}} a^+_\ell - \sum_{\ell=n_k+1}^{n_{k+1}} a^-_\ell \right) \right. \\
\left. + \sum_{\ell=m_p+1}^{n_p+1} a^+_\ell - \sum_{\ell=n_{p+1}}^{n} a^-_\ell < c\right\}.
\]
Consider the series
\[
(4.13) \quad a^+_1 + a^+_2 + \cdots + a^+_{m_1} - a^-_1 - a^-_2 - \cdots - a^-_{n_1} \\
+ a^+_{m_1+1} + a^+_{m_1+2} + \cdots + a^+_{m_2} - a^-_{n_1+1} - a^-_{n_1+2} - \cdots - a^-_{n_2} \\
+ a^+_{n_2+1} + a^+_{n_2+2} + \cdots + a^+_{n_3} - a^-_{n_2+1} - a^-_{n_2+2} - \cdots - a^-_{n_3} \\
+ \cdots
\]
It is clear this series is a rearrangement of \( \sum_{n=1}^{\infty} a_n \) and the way in which \( m_p \) and \( n_p \) were chosen guarantee that
\[
0 < \sum_{k=0}^{p-1} \left( \sum_{\ell=m_k+1}^{m_{k+1}} a^+_\ell - \sum_{\ell=n_k+1}^{n_{k+1}} a^-_\ell + \sum_{k=m_p+1}^{m_p} a^+_k \right) - c \leq a^+_{m_p}
\]
and
\[
0 < c - \sum_{k=0}^{p} \left( \sum_{\ell=m_k+1}^{m_{k+1}} a^+_\ell - \sum_{\ell=n_k+1}^{n_{k+1}} a^-_\ell \right) \leq a^-_{n_p}
\]
Since both \( a^+_{m_p} \to 0 \) and \( a^-_{n_p} \to 0 \), the result follows from the Squeeze Theorem.

The argument when \( c \) is infinite is left as Exercise 4.31. □
A moral to take from all this is that absolutely convergent series are robust and conditionally convergent series are fragile. Absolutely convergent series can be sliced and diced and mixed with careless abandon without getting surprising results. If conditionally convergent series are not handled with care, the results can be quite unexpected.

5. Exercises

4.1. Prove Theorem 4.4.

4.2. If \( \sum_{n=1}^{\infty} a_n \) is a convergent positive series, then does \( \sum_{n=1}^{\infty} \frac{1}{1+a_n} \) converge?

4.3. The series \( \sum_{n=1}^{\infty} (a_n - a_{n+1}) \) converges iff the sequence \( a_n \) converges.

4.4. Prove or give a counter example: If \( \sum |a_n| \) converges, then \( na_n \rightarrow 0 \).

4.5. If the series \( a_1 + a_2 + a_3 + \cdots \) converges to \( S \), then so does

\[
(4.14) \quad a_1 + 0 + a_2 + 0 + 0 + a_3 + 0 + 0 + a_4 + \cdots.
\]

4.6. If \( \sum_{n=1}^{\infty} a_n \) converges and \( b_n \) is a bounded monotonic sequence, then \( \sum_{n=1}^{\infty} a_n b_n \) converges.

4.7. Let \( x_n \) be a sequence with range \( \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} \). Prove that \( \sum_{n=1}^{\infty} x_n 10^{-n} \) converges and its sum is in the interval \( [0, 1] \).

4.8. Write 6.17272727272\cdots as a fraction.

4.9. Prove the ratio test by setting \( p_n = 1 \) for all \( n \) in Kummer's test.

4.10. Consider the series

\[
1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \cdots = 4.
\]

Show that the ratio test is inconclusive for this series, but the root test gives a positive answer.

4.11. Does \( \sum_{n=2}^{\infty} \frac{1}{n(ln n)^2} \) converge?

4.12. Does

\[
\frac{1}{3} + \frac{1 \times 2}{3 \times 5} + \frac{1 \times 2 \times 3}{3 \times 5 \times 7} + \frac{1 \times 2 \times 3 \times 4}{3 \times 5 \times 7 \times 9} + \cdots
\]

converge?
4.13. For what values of $p$ does
\[
\left( \frac{1}{2} \right)^p + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^p + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^p + \cdots
\]
converge?

4.14. Find sequences $a_n$ and $b_n$ satisfying:
(a) $a_n > 0, \forall n \in \mathbb{N}$ and $a_n \to 0$;
(b) $B_n = \sum_{k=1}^{n} b_k$ is a bounded sequence; and,
(c) $\sum_{n=1}^{\infty} a_n b_n$ diverges.

4.15. Let $a_n$ be a sequence such that $a_{2n} \to A$ and $a_{2n} - a_{2n-1} \to 0$. Then $a_n \to A$.

4.16. Prove Bertrand's test, Theorem 4.15.

4.17. Let $\sum a_n$ be a positive series. Prove that $\sum a_n$ converges if and only if there is a sequence of positive numbers $p_n$ and $\alpha > 0$ such that
\[
\lim_{n \to \infty} p_n \frac{a_n}{a_{n+1}} - p_{n+1} = \alpha.
\]
(Hint: If $s = \sum a_n$ and $s_n = \sum_{k=1}^{n} a_k$, then let $p_n = (s - s_n)/a_n$.)

4.18. Prove that $\sum_{n=0}^{\infty} x^n/n!$ converges for all $x \in \mathbb{R}$.

4.19. Find all values of $x$ for which $\sum_{k=0}^{\infty} k^2(x + 3)^k$ converges.

4.20. For what values of $x$ does the series
\[
\sum_{n=1}^{\infty} \frac{(-1)^{n+1} x^{2n-1}}{2n-1}
\]
converge?

4.21. For what values of $x$ does $\sum_{n=1}^{\infty} \frac{(x + 3)^n}{n4^n}$ converge absolutely, converge conditionally or diverge?

4.22. For what values of $x$ does $\sum_{n=1}^{\infty} \frac{n + 6}{n^2(x - 1)^n}$ converge absolutely, converge conditionally or diverge?

4.23. For what positive values of $\alpha$ does $\sum_{n=1}^{\infty} a^n n^\alpha$ converge?

4.24. Prove that $\sum \cos \frac{n\pi}{3} \sin \frac{\pi}{n}$ converges.
4.25. For a series $\sum_{k=1}^{\infty} a_n$ with partial sums $s_n$, define

$$\sigma_n = \frac{1}{n} \sum_{k=1}^{n} s_n.$$  

Prove that if $\sum_{k=1}^{\infty} a_n = s$, then $\sigma_n \to s$. Find an example where $\sigma_n$ converges, but $\sum_{k=1}^{\infty} a_n$ does not. (If $\sigma_n$ converges, the sequence is said to be Cesàro summable.)

4.26. If $a_n$ is a sequence with a subsequence $b_n$, then $\sum_{n=1}^{\infty} b_n$ is a subseries of $\sum_{n=1}^{\infty} a_n$. Prove that if every subseries of $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

4.27. If $\sum_{n=1}^{\infty} a_n$ is a convergent positive series, then so is $\sum_{n=1}^{\infty} a_n^2$. Give an example to show the converse is not true.

4.28. Prove or give a counter example: If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} a_n^2$ converges.

4.29. For what positive values of $\alpha$ does $\sum_{n=1}^{\infty} a_n^\alpha$ converge?

4.30. If $a_n \geq 0$ for all $n \in \mathbb{N}$ and there is a $p > 1$ such that $\lim_{n \to \infty} n^p a_n$ exists and is finite, then $\sum_{n=1}^{\infty} a_n$ converges. Is this true for $p = 1$?

4.31. Finish the proof of Theorem 4.23.

4.32. Leonhard Euler started with the equation

$$\frac{x}{x-1} + \frac{x}{1-x} = 0,$$

transformed it to

$$\frac{1}{1-1/x} + \frac{x}{1-x} = 0,$$

and then used geometric series to write it as

$$\cdots + \frac{1}{x^2} + \frac{1}{x} + 1 + x + x^2 + x^3 + \cdots = 0.$$  

(4.23)

Show how Euler did his calculation and find his mistake.

4.33. Let $\sum a_n$ be a conditionally convergent series and $c \in \mathbb{R} \cup \{-\infty, \infty\}$. There is a sequence $b_n$ such that $|b_n| = 1$ for all $n \in \mathbb{N}$ and $\sum a_n b_n = c$. 

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http://math.louisville.edu/~lee/ira