CHAPTER 6

Limits of Functions

1. Basic Definitions

**Definition 6.1.** Let \( D \subset \mathbb{R} \), \( x_0 \) be a limit point of \( D \) and \( f : D \to \mathbb{R} \). The limit of \( f(x) \) at \( x_0 \) is \( L \), if for each \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that when \( x \in D \) with \( 0 < |x-x_0| < \delta \), then \( |f(x) - L| < \varepsilon \). When this is the case, we write \( \lim_{x \to x_0} f(x) = L \).

It should be noted that the limit of \( f \) at \( x_0 \) is determined by the values of \( f \) near \( x_0 \) and not at \( x_0 \). In fact, \( f \) need not even be defined at \( x_0 \).

**Figure 1.** This figure shows a way to think about the limit. The graph of \( f \) must not leave the top or bottom of the box \((x_0 - \delta, x_0 + \delta) \times (L - \varepsilon, L + \varepsilon)\), except possibly the point \((x_0, f(x_0))\).

A useful way of rewording the definition is to say that \( \lim_{x \to x_0} f(x) = L \) iff for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\} \) implies \( f(x) \in (L - \varepsilon, L + \varepsilon) \). This can also be succinctly stated as

\[
\forall \varepsilon > 0 \exists \delta > 0 \left( f( (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\} ) \subset (L - \varepsilon, L + \varepsilon) \right).
\]

**Example 6.1.** If \( f(x) = c \) is a constant function and \( x_0 \in \mathbb{R} \), then for any positive numbers \( \varepsilon \) and \( \delta \),

\[
x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{x_0\} \Rightarrow |f(x) - c| = |c - c| = 0 < \varepsilon.
\]

This shows the limit of every constant function exists at every point, and the limit is just the value of the function.
CHAPTER 6. LIMITS OF FUNCTIONS

EXAMPLE 6.2. Let \( f(x) = x, x_0 \in \mathbb{R} \), and \( \varepsilon = \delta > 0 \). Then
\[
x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{ x_0 \} \Rightarrow |f(x) - x_0| = |x - x_0| < \delta = \varepsilon.
\]
This shows that the identity function has a limit at every point and its limit is just the value of the function at that point.

EXAMPLE 6.3. Let \( f(x) = \frac{2x^2 - 8}{x - 2} \). In this case, the implied domain of \( f \) is \( D = \mathbb{R} \setminus \{2\} \). We claim that \( \lim_{x \to -2} f(x) = 8 \).

To see this, let \( \varepsilon > 0 \) and choose \( \delta \in (0, \varepsilon/2) \). If \( 0 < |x - 2| < \delta \), then
\[
|f(x) - 8| = \left| \frac{2x^2 - 8}{x - 2} - 8 \right| = \left| 2(x + 2) - 8 \right| = 2|x - 2| < \varepsilon.
\]

EXAMPLE 6.4. Let \( f(x) = \sqrt{x + 1} \). Then the implied domain of \( f \) is \( D = [-1, \infty) \). We claim that \( \lim_{x \to -1} f(x) = 0 \).

To see this, let \( \varepsilon > 0 \) and choose \( \delta \in (0, \varepsilon^2) \). If \( 0 < x - (-1) = x + 1 < \delta \), then
\[
|f(x) - 0| = \sqrt{x + 1} < \sqrt{\delta} < \sqrt{\varepsilon^2} = \varepsilon.
\]

EXAMPLE 6.5. If \( f(x) = |x|/x \) for \( x \neq 0 \), then \( \lim_{x \to 0} f(x) \) does not exist. (See Figure 3.) To see this, suppose \( \lim_{x \to 0} f(x) = L \), \( \varepsilon = 1 \) and \( \delta > 0 \). If \( L \geq 0 \) and \( -\delta < x < 0 \), then \( f(x) = -1 < L - \varepsilon \). If \( L < 0 \) and \( 0 < x < \delta \), then \( f(x) = 1 > L + \varepsilon \). These inequalities show for any \( L \) and every \( \delta > 0 \), there is an \( x \) with \( 0 < |x| < \delta \) such that \( |f(x) - L| > \varepsilon \).
There is an obvious similarity between the definition of limit of a sequence and limit of a function. The following theorem makes this similarity explicit, and gives another way to prove facts about limits of functions.

**Theorem 6.2.** Let \( f : D \to \mathbb{R} \) and \( x_0 \) be a limit point of \( D \). \( \lim_{x \to x_0} f(x) = L \) iff whenever \( x_n \to x_0 \) is a sequence from \( D \setminus \{ x_0 \} \) such that \( x_n \to x_0 \), then \( f(x_n) \to L \).

**Proof.** (\( \Rightarrow \)) Suppose \( \lim_{x \to x_0} f(x) = L \) and \( x_n \) is a sequence from \( D \setminus \{ x_0 \} \) such that \( x_n \to x_0 \). Let \( \varepsilon > 0 \). There exists a \( \delta > 0 \) such that \( |f(x) - L| < \varepsilon \) whenever \( x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{ x_0 \} \). Since \( x_n \to x_0 \), there is an \( N \in \mathbb{N} \) such that for all \( n \geq N \) implies \( 0 < |x_n - x_0| < \delta \). In this case, \( |f(x_n) - L| < \varepsilon \), showing \( f(x_n) \to L \).

(\( \Leftarrow \)) Suppose whenever \( x_n \) is a sequence from \( D \setminus \{ x_0 \} \) such that \( x_n \to x_0 \), then \( f(x_n) \to L \), but \( \lim_{x \to x_0} f(x) \neq L \). Then there exists an \( \varepsilon > 0 \) such that for all \( \delta > 0 \) there is an \( x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{ x_0 \} \) such that \( |f(x) - L| \geq \varepsilon \). In particular, for each \( n \in \mathbb{N} \), there must exist \( x_n \in (x_0 - 1/n, x_0 + 1/n) \cap D \setminus \{ x_0 \} \) such that \( |f(x_n) - L| \geq \varepsilon \). Since \( x_n \to x_0 \), this is a contradiction. Therefore, \( \lim_{x \to x_0} f(x) = L \).

Theorem 6.2 is often used to show a limit doesn’t exist. Suppose we want to show \( \lim_{x \to x_0} f(x) \) doesn’t exist. There are two strategies: find a sequence \( x_n \to x_0 \) such that \( f(x_n) \) has no limit; or, find two sequences \( y_n \to x_0 \) and \( z_n \to x_0 \) such that \( f(y_n) \) and \( f(z_n) \) converge to different limits. Either way, the theorem shows \( \lim_{x \to x_0} f(x) \) fails to exist.

In Example 6.5, we could choose \( x_n = (-1)^n \) so \( f(x_n) \) oscillates between -1 and 1. Or, we could choose \( y_n = 1/n = -z_n \) so \( f(x_n) \to 1 \) and \( f(z_n) \to -1 \).

**Example 6.6.** Let \( f(x) = \sin(1/x) \), \( a_n = \frac{1}{n\pi} \) and \( b_n = \frac{2}{(3n+1)\pi} \). Then \( a_n \downarrow 0 \), \( b_n \downarrow 0 \), \( f(a_n) = 0 \) and \( f(b_n) = 1 \) for all \( n \in \mathbb{N} \). An application of Theorem 6.2 shows \( \lim_{x \to 0} f(x) \) does not exist. (See Figure 4.)

**Theorem 6.3 (Squeeze Theorem).** Suppose \( f \), \( g \) and \( h \) are all functions defined on \( D \subseteq \mathbb{R} \) with \( f(x) \leq g(x) \leq h(x) \) for all \( x \in D \). If \( x_0 \) is a limit point of \( D \) and \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0} h(x) = L \), then \( \lim_{x \to x_0} g(x) = L \).

**Proof.** Let \( x_n \) be any sequence from \( D \setminus \{ x_0 \} \) such that \( x_n \to x_0 \). According to Theorem 6.2, both \( f(x_n) \to L \) and \( h(x_n) \to L \). Since \( f(x_n) \leq g(x_n) \leq h(x_n) \),
an application of the Sandwich Theorem for sequences shows \( g(x_n) \to L \). Now, another use of Theorem 6.2 shows \( \lim_{x \to x_0} g(x) = L \). □

**Example 6.7.** Let \( f(x) = x \sin(1/x) \). Since \( -1 \leq \sin(1/x) \leq 1 \) when \( x \neq 0 \), we see that \( -x \leq x \sin(1/x) \leq x \) for \( x \neq 0 \). Since \( \lim_{x \to 0} x = \lim_{x \to 0} -x = 0 \), Theorem 6.3 implies \( \lim_{x \to 0} x \sin(1/x) = 0 \). (See Figure 5.)

**Theorem 6.4.** Suppose \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \) and \( x_0 \) is a limit point of \( D \). If \( \lim_{x \to x_0} f(x) = L \) and \( \lim_{x \to x_0} g(x) = M \), then

(a) \( \lim_{x \to x_0} (f + g)(x) = L + M \),

(b) \( \lim_{x \to x_0} (af)(x) = aL \), \( \forall a \in \mathbb{R} \),

(c) \( \lim_{x \to x_0} (fg)(x) = LM \), and

(d) \( \lim_{x \to x_0} (1/f)(x) = 1/L \), as long as \( L \neq 0 \).

**Proof.** Suppose \( a_n \) is a sequence from \( D \setminus \{x_0\} \) converging to \( x_0 \). Then Theorem 6.2 implies \( f(a_n) \to L \) and \( g(a_n) \to M \). (a)-(d) follow at once from the corresponding properties for sequences. □

**Example 6.8.** Let \( f(x) = 3x + 2 \). If \( g_1(x) = 3 \), \( g_2(x) = x \) and \( g_3(x) = 2 \), then \( f(x) = g_1(x)g_2(x) + g_3(x) \). Examples 6.1 and 6.2 along with parts (a) and (c) of Theorem 6.4 immediately show that for every \( x \in \mathbb{R} \), \( \lim_{x \to x_0} f(x) = f(x_0) \).

In the same manner as Example 6.8, it can be shown for every rational function \( f(x) \), that \( \lim_{x \to x_0} f(x) = f(x_0) \) whenever \( f(x_0) \) exists.
2. Unilateral Limits

**Definition 6.5.** Let \( f : D \to \mathbb{R} \) and \( x_0 \) be a limit point of \((-\infty, x_0) \cap D\). \( f \) has \( L \) as its *left-hand limit* at \( x_0 \) if for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( f((x_0 - \delta, x_0) \cap D) \subset (L - \varepsilon, L + \varepsilon) \). In this case, we write \( \lim_{x \to x_0} f(x) = L \).

Let \( f : D \to \mathbb{R} \) and \( x_0 \) be a limit point of \( D \cap (x_0, \infty) \). \( f \) has \( L \) as its *right-hand limit* at \( x_0 \) if for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( f(D \cap (x_0, x_0 + \delta)) \subset (L - \varepsilon, L + \varepsilon) \). In this case, we write \( \lim_{x \to x_0, x > 0} f(x) = L \).

These are called the *unilateral* or *one-sided* limits of \( f \) at \( x_0 \). When they are different, the graph of \( f \) is often said to have a “jump” at \( x_0 \), as in the following example.

**Example 6.9.** As in Example 6.5, let \( f(x) = |x|/x \). Then \( \lim_{x \to 0} f(x) = 1 \) and \( \lim_{x \to 0} f(x) = -1 \). (See Figure 3.)

In parallel with Theorem 6.2, the one-sided limits can also be reformulated in terms of sequences.

**Theorem 6.6.** Let \( f : D \to \mathbb{R} \) and \( x_0 \).

(a) Let \( x_0 \) be a limit point of \( D \cap (x_0, \infty) \). \( \lim_{x \to x_0} f(x) = L \) iff whenever \( x_n \) is a sequence from \( D \cap (x_0, \infty) \) such that \( x_n \to x_0 \), then \( f(x_n) \to L \).

(b) Let \( x_0 \) be a limit point of \((-\infty, x_0) \cap D\). \( \lim_{x \to x_0} f(x) = L \) iff whenever \( x_n \) is a sequence from \((-\infty, x_0) \cap D \) such that \( x_n \to x_0 \), then \( f(x_n) \to L \).

The proof of Theorem 6.6 is similar to that of Theorem 6.2 and is left to the reader.

**Theorem 6.7.** Let \( f : [a, b] \to \mathbb{R} \) and \( x_0 \) be a limit point of \( D \).

\[
\lim_{x \to x_0} f(x) = L \iff \lim_{x \to x_0^{-}} f(x) = L = \lim_{x \to x_0^{+}} f(x)
\]

**Proof.** This proof is left as an exercise.

**Theorem 6.8.** If \( f : (a, b) \to \mathbb{R} \) is monotone, then both unilateral limits of \( f \) exist at every point of \((a, b)\).

**Proof.** To be specific, suppose \( f \) is increasing and \( x_0 \in (a, b) \). Let \( \varepsilon > 0 \) and \( L = \text{lub} \{ f(x) : a < x < x_0 \} \). According to Corollary 2.19, there must exist an \( x \in (a, x_0) \) such that \( L - \varepsilon < f(x) \leq L \). Define \( \delta = x_0 - x \). If \( y \in (x_0 - \delta, x_0) \), then \( L - \varepsilon < f(x) \leq f(y) \leq L \). This shows \( \lim_{x \to x_0} f(x) = L \).

The proof that \( \lim_{x \to x_0} f(x) \) exists is similar.

To handle the case when \( f \) is decreasing, consider \(-f\) instead of \( f \).

3. Continuity

**Definition 6.9.** Let \( f : D \to \mathbb{R} \) and \( x_0 \in D \). \( f \) is *continuous at* \( x_0 \) if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that when \( x \in D \) with \(|x - x_0| < \delta\), then \(|f(x) - f(x_0)| < \varepsilon\). The set of all points at which \( f \) is continuous is denoted \( C(f) \).
Several useful ways of rephrasing this are contained in the following theorem. They are analogous to the similar statements made about limits. Proofs are left to the reader.

**Theorem 6.10.** Let \( f : D \to \mathbb{R} \) and \( x_0 \in D \). The following statements are equivalent.

(a) \( x_0 \in C(f) \),

(b) For all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
x \in (x_0 - \delta, x_0 + \delta) \cap D \implies f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon),
\]

(c) For all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that

\[
f((x_0 - \delta, x_0 + \delta) \cap D) \subset (f(x_0) - \varepsilon, f(x_0) + \varepsilon).
\]

**Example 6.10.** Define

\[
f(x) = \begin{cases} \frac{2x^2 - 8}{x - 2}, & x \neq 2 \\ 8, & x = 2 \end{cases}
\]

It follows from Example 6.3 that 2 \( \in C(f) \).

There is a subtle difference between the treatment of the domain of the function in the definitions of limit and continuity. In the definition of limit, the “target point,” \( x_0 \) is required to be a limit point of the domain, but not actually be an element of the domain. In the definition of continuity, \( x_0 \) must be in the domain of the function, but does not have to be a limit point. To see a consequence of this difference, consider the following example.

**Example 6.11.** If \( f : \mathbb{Z} \to \mathbb{R} \) is an arbitrary function, then \( C(f) = \mathbb{Z} \). To see this, let \( n_0 \in \mathbb{Z}, \varepsilon > 0 \) and \( \delta = 1 \). If \( x \in \mathbb{Z} \) with \( |x - n_0| < \delta \), then \( x = n_0 \). It follows that \( |f(x) - f(n_0)| = 0 < \varepsilon \), so \( f \) is continuous at \( n_0 \).

\(^2\) Calculus books often use the notation \( \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^+} - f(x) \) and \( \lim_{x \to x_0} f(x) = \lim_{x \to x_0^+} \lim_{x \to x_0} f(x) \).
This leads to the following theorem.

**Theorem 6.11.** Let \( f : D \to \mathbb{R} \) and \( x_0 \in D \). If \( x_0 \) is a limit point of \( D \), then \( x_0 \in C(f) \) iff \( \lim_{x \to x_0} f(x) = f(x_0) \). If \( x_0 \) is an isolated point of \( D \), then \( x_0 \in C(f) \).

**Proof.** If \( x_0 \) is isolated in \( D \), then there is an \( \delta > 0 \) such that \( (x_0 - \delta, x_0 + \delta) \cap D = \{ x_0 \} \). For any \( \varepsilon > 0 \), the definition of continuity is satisfied with this \( \delta \).

Next, suppose \( x_0 \in D \).

The definition of continuity says that \( f \) is continuous at \( x_0 \) iff for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that when \( x \in (x_0 - \delta, x_0 + \delta) \cap D \), then \( f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \).

The definition of limit says that \( \lim_{x \to x_0} f(x) = f(x_0) \) iff for all \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that when \( x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{ x_0 \} \), then \( f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \).

Comparing these two definitions, it is clear that \( x_0 \in C(f) \) implies

\[
\lim_{x \to x_0} f(x) = f(x_0).
\]

On the other hand, suppose \( \lim_{x \to x_0} f(x) = f(x_0) \) and \( \varepsilon > 0 \). Choose \( \delta \) according to the definition of limit. When \( x \in (x_0 - \delta, x_0 + \delta) \cap D \setminus \{ x_0 \} \), then \( f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \). It follows from this that when \( x = x_0 \), then \( f(x) - f(x_0) = f(x_0) - f(x_0) = 0 < \varepsilon \). Therefore, when \( x \in (x_0 - \delta, x_0 + \delta) \cap D \), then \( f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \), and \( x_0 \in C(f) \), as desired.

**Example 6.12.** If \( f(x) = c \), for some \( c \in \mathbb{R} \), then Example 6.1 and Theorem 6.11 show that \( f \) is continuous at every point.

**Example 6.13.** If \( f(x) = x \), then Example 6.2 and Theorem 6.11 show that \( f \) is continuous at every point.

**Corollary 6.12.** Let \( f : D \to \mathbb{R} \) and \( x_0 \in D \). \( x_0 \in C(f) \) iff whenever \( x_n \) is a sequence from \( D \) with \( x_n \to x_0 \), then \( f(x_n) \to f(x_0) \).

**Proof.** Combining Theorem 6.11 with Theorem 6.2 shows this to be true.

**Example 6.14 (Dirichlet Function).** Suppose

\[
f(x) = \begin{cases} 
1, & x \in \mathbb{Q} \\
0, & x \notin \mathbb{Q}.
\end{cases}
\]

For each \( x \in \mathbb{Q} \), there is a sequence of irrational numbers converging to \( x \), and for each \( y \in \mathbb{Q}^c \) there is a sequence of rational numbers converging to \( y \). Corollary 6.12 shows \( C(f) = \emptyset \).

**Example 6.15 (Salt and Pepper Function).** Since \( \mathbb{Q} \) is a countable set, it can be written as a sequence, \( \mathbb{Q} = \{ q_n : n \in \mathbb{N} \} \). Define

\[
f(x) = \begin{cases} 
1/n, & x = q_n, \\
0, & x \in \mathbb{Q}^c.
\end{cases}
\]

If \( x \in \mathbb{Q} \), then \( x = q_n \), for some \( n \) and \( f(x) = 1/n > 0 \). There is a sequence \( x_n \) from \( \mathbb{Q}^c \) such that \( x_n \to x \) and \( f(x_n) = 0 \neq f(x) = 1/n \). Therefore \( C(f) \cap \mathbb{Q} = \emptyset \).
On the other hand, let \( x \in \mathbb{Q}^c \) and \( \varepsilon > 0 \). Choose \( N \in \mathbb{N} \) large enough so that \( 1/N < \varepsilon \) and let \( \delta = \min\{|x - q_n| : 1 \leq n \leq N\} \). If \( |x - y| < \delta \), there are two cases to consider. If \( y \in \mathbb{Q}^c \), then \( |f(y) - f(x)| = |0 - 0| = 0 < \varepsilon \). If \( y \in \mathbb{Q} \), then the choice of \( \delta \) guarantees \( y = q_n \) for some \( n > N \). In this case, \( |f(y) - f(x)| = f(y) = f(q_n) = 1/n < 1/N < \varepsilon \). Therefore, \( x \in C(f) \).

This shows that \( C(f) = \mathbb{Q}^c \).

It is a consequence of the Baire category theorem that there is no function \( f \) such that \( C(f) = \mathbb{Q} \). Proving this would take us too far afield.

The following theorem is an almost immediate consequence of Theorem 6.4.

**Theorem 6.13.** Let \( f : D \to \mathbb{R} \) and \( g : D \to \mathbb{R} \). If \( x_0 \in C(f) \cap C(g) \), then

(a) \( x_0 \in C(f + g) \),
(b) \( x_0 \in C(\alpha f) \), \( \forall \alpha \in \mathbb{R} \),
(c) \( x_0 \in C(fg) \), and
(d) \( x_0 \in C(f/g) \) when \( g(x_0) \neq 0 \).

**Corollary 6.14.** If \( f \) is a rational function, then \( f \) is continuous at each point of its domain.

**Proof.** This is a consequence of Examples 6.12 and 6.13 used with Theorem 6.13.

**Theorem 6.15.** Suppose \( f : D_f \to \mathbb{R} \) and \( g : D_g \to \mathbb{R} \) such that \( f(D_f) \subseteq D_g \). If there is an \( x_0 \in C(f) \) such that \( f(x_0) \in C(g) \), then \( x_0 \in C(g \circ f) \).

**Proof.** Let \( \varepsilon > 0 \) and choose \( \delta_1 > 0 \) such that

\[
g((f(x_0) - \delta_1, f(x_0) + \delta_1) \cap D_g) \subseteq (g \circ f(x_0) - \varepsilon, g \circ f(x_0) + \varepsilon).
\]

Choose \( \delta_2 > 0 \) such that

\[
f((x_0 - \delta_2, x_0 + \delta_2) \cap D_f) \subseteq (f(x_0) - \delta_1, f(x_0) + \delta_1).
\]

Then

\[
g \circ f((x_0 - \delta_2, x_0 + \delta_2) \cap D_f) \subseteq g((f(x_0) - \delta_1, f(x_0) + \delta_1) \cap D_g)
\]

\[
\subseteq (g \circ f(x_0) - \delta_2, g \circ f(x_0) + \delta_2) \cap D_f).
\]

Since this shows Theorem 6.10(c) is satisfied at \( x_0 \) with the function \( g \circ f \), it follows that \( x_0 \in C(g \circ f) \).

**Example 6.16.** If \( f(x) = \sqrt{x} \) for \( x \geq 0 \), then according to Problem 6.8, \( C(f) = [0, \infty) \). Theorem 6.15 shows \( f \circ f(x) = \sqrt{\sqrt{x}} \) is continuous on \( [0, \infty) \).

In similar way, it can be shown by induction that \( f(x) = x^{m/2^n} \) is continuous on \( [0, \infty) \) for all \( m, n \in \mathbb{Z} \).

**4. Unilateral Continuity**

**Definition 6.16.** Let \( f : D \to \mathbb{R} \) and \( x_0 \in D \). \( f \) is **left-continuous** at \( x_0 \) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( f((x_0 - \delta, x_0] \cap D) \subseteq (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \).

Let \( f : D \to \mathbb{R} \) and \( x_0 \in D \). \( f \) is **right-continuous** at \( x_0 \) if for every \( \varepsilon > 0 \) there is a \( \delta > 0 \) such that \( f([x_0, x_0 + \delta) \cap D) \subseteq (f(x_0) - \varepsilon, f(x_0) + \varepsilon) \).
Example 6.17. Let the floor function be
\[ [x] = \max\{n \in \mathbb{Z} : n \leq x \} \]
and the ceiling function be
\[ [x] = \min\{n \in \mathbb{Z} : n \geq x \}. \]
The floor function is right-continuous, but not left-continuous at each integer, and the ceiling function is left-continuous, but not right-continuous at each integer.

Theorem 6.17. Let \( f : D \to \mathbb{R} \) and \( x_0 \in D \). \( x_0 \in C(f) \) iff \( f \) is both right and left-continuous at \( x_0 \).

Proof. The proof of this theorem is left as an exercise. \( \square \)

According to Theorem 6.7, when \( f \) is monotone on an interval \((a, b)\), the unilateral limits of \( f \) exist at every point. In order for such a function to be continuous at \( x_0 \in (a, b) \), it must be the case that
\[ \lim_{x \to x_0^+} f(x) = f(x_0) = \lim_{x \to x_0^-} f(x). \]
If either of the two equalities is violated, the function is not continuous at \( x_0 \).

In the case, when \( \lim_{x \to x_0^+} f(x) \neq \lim_{x \to x_0^-} f(x) \), it is said that a jump discontinuity occurs at \( x_0 \).

Example 6.18. The function
\[ f(x) = \begin{cases} \frac{|x|}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases} \]
has a jump discontinuity at \( x = 0 \).

In the case when \( \lim_{x \to x_0^+} f(x) = \lim_{x \to x_0^-} f(x) \neq f(x_0) \), it is said that \( f \) has a removable discontinuity at \( x_0 \). The discontinuity is called “removable” because in this case, the function can be made continuous at \( x_0 \) by merely redefining its value at the single point, \( x_0 \), to be the value of the two one-sided limits.

Example 6.19. The function \( f(x) = \frac{x^2-4}{x-2} \) is not continuous at \( x = 2 \) because 2 is not in the domain of \( f \). Since \( \lim_{x \to 2^-} f(x) = 4 \), if the domain of \( f \) is extended to include 2 by setting \( f(2) = 4 \), then this extended \( f \) is continuous everywhere. (See Figure 7.)

Theorem 6.18. If \( f : (a, b) \to \mathbb{R} \) is monotone, then \((a, b) \setminus C(f)\) is countable.

Proof. In light of the discussion above and Theorem 6.7, it is apparent that the only types of discontinuities \( f \) can have are jump discontinuities.

To be specific, suppose \( f \) is increasing and \( x_0, y_0 \in (a, b) \setminus C(f) \) with \( x_0 < y_0 \).

In this case, the fact that \( f \) is increasing implies
\[ \lim_{x \to x_0^+} f(x) \leq \lim_{x \to y_0^-} f(x) \leq \lim_{x \to y_0^+} f(x) \leq \lim_{x \to x_0^-} f(x). \]

This implies that for any two \( x_0, y_0 \in (a, b) \setminus C(f) \), there are disjoint open intervals, \( I_{x_0} = (\lim_{x \to x_0^+} f(x), \lim_{x \to x_0^-} f(x)) \) and \( I_{y_0} = (\lim_{x \to y_0^-} f(x), \lim_{x \to y_0^+} f(x)) \). For each
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\[ f(x) = \frac{x^2 - 4}{x-2} \]

**Figure 7.** The function from Example 6.19. Note that the graph is a line with one “hole” in it. Plugging up the hole removes the discontinuity.

Let \( x \in (a, b) \setminus C(f) \), choose \( q_x \in I_x \cap \mathbb{Q} \). Because of the pairwise disjointness of the intervals \( \{I_x : x \in (a, b) \setminus C(f)\} \), this defines an bijection between \( (a, b) \setminus C(f) \) and a subset of \( \mathbb{Q} \). Therefore, \( (a, b) \setminus C(f) \) must be countable.

A similar argument holds for a decreasing function.

Theorem 6.18 implies that a monotone function is continuous at “nearly every” point in its domain. Characterizing the points of discontinuity as countable is the best that can be hoped for, as seen in the following example.

**Example 6.20.** Let \( D = \{d_n : n \in \mathbb{N}\} \) be a countable set and define \( J_x = \{n : d_n < x\} \). The function

\[
 f(x) = \begin{cases} 
 0, & J_x = \emptyset \\
 \sum_{n \in J_x} \frac{1}{2^n}, & J_x \neq \emptyset 
\end{cases}
\]

is increasing and \( C(f) = D^c \). The proof of this statement is left as Exercise 6.9.

5. Continuous Functions

Up until now, continuity has been considered as a property of a function at a point. There is much that can be said about functions continuous everywhere.

**Definition 6.19.** Let \( f : D \to \mathbb{R} \) and \( A \subset D \). We say \( f \) is continuous on \( A \) if \( A \subset C(f) \). If \( D = C(f) \), then \( f \) is continuous.

Continuity at a point is, in a sense, a metric property of a function because it measures relative distances between points in the domain and image sets. Continuity on a set becomes more of a topological property, as shown by the next few theorems.

**Theorem 6.20.** \( f : D \to \mathbb{R} \) is continuous iff whenever \( G \) is open in \( \mathbb{R} \), then \( f^{-1}(G) \) is relatively open in \( D \).
Section 5: Continuous Functions

**Theorem 6.21.** If \( f \) is continuous on a compact set \( K \), then \( f(K) \) is compact.

**Proof.** Let \( \mathcal{O} \) be an open cover of \( f(K) \) and \( \mathcal{F} = \{ f^{-1}(G) : G \in \mathcal{O} \} \). By Theorem 6.20, \( \mathcal{F} \) is a collection of sets which are relatively open in \( K \). Since \( \mathcal{F} \) covers \( K \), \( \mathcal{F} \) is an open cover of \( K \). Using the fact that \( K \) is compact, we can choose a finite subcover of \( K \) from \( \mathcal{F} \), say \( \{ G_1, G_2, \ldots, G_n \} \). Then \( \{ H_1, H_2, \ldots, H_n \} \in \mathcal{O} \) such that \( f^{-1}(H_k) = G_k \) for \( 1 \leq k \leq n \). Thus, \( \{ H_1, H_2, \ldots, H_3 \} \) is a subcover of \( f(K) \) from \( \mathcal{O} \).

Several of the standard calculus theorems giving properties of continuous functions are consequences of Corollary 6.21. In a calculus course, \( K \) is usually a compact interval, \([a, b]\).

**Corollary 6.22.** If \( f : K \rightarrow \mathbb{R} \) is continuous and \( K \) is compact, then \( f \) is bounded.

**Proof.** By Theorem 6.21, \( f(K) \) is compact. Now, use the Bolzano-Weierstrass theorem to conclude \( f \) is bounded.

**Corollary 6.23** (Maximum Value Theorem). If \( f : K \rightarrow \mathbb{R} \) is continuous and \( K \) is compact, then there are \( m, M \in K \) such that \( f(m) \leq f(x) \leq f(M) \) for all \( x \in K \).

**Proof.** According to Theorem 6.21 and the Bolzano-Weierstrass theorem, \( f(K) \) is closed and bounded. Because of this, \( \text{glb} f(K) \in f(K) \) and \( \text{lub} f(K) \in f(K) \). It suffices to choose \( m \in f^{-1}(\text{glb} f(K)) \) and \( M \in f^{-1}(\text{lub} f(K)) \).

**Corollary 6.24.** If \( f : K \rightarrow \mathbb{R} \) is continuous and invertible and \( K \) is compact, then \( f^{-1} : f(K) \rightarrow K \) is continuous.

**Proof.** Let \( G \) be open in \( K \). According to Theorem 6.20, it suffices to show \( f(G) \) is open in \( f(K) \).

To do this, note that \( K \setminus G \) is compact, so by Theorem 6.21, \( f(K \setminus G) \) is compact, and therefore closed. Because \( f \) is injective, \( f(G) = f(K) \setminus f(K \setminus G) \). This shows \( f(G) \) is open in \( f(K) \).
6.25. If $f$ is continuous on an interval $I$, then $f(I)$ is an interval.

**Proof.** If $f(I)$ is not connected, there must exist two disjoint open sets, $U$ and $V$, such that $f(I) \subset U \cup V$ and $f(I) \cap U \neq \emptyset \neq f(I) \cap V$. In this case, Theorem 6.20 implies $f^{-1}(U)$ and $f^{-1}(V)$ are both open. They are clearly disjoint and $f^{-1}(U) \cap I \neq \emptyset \neq f^{-1}(V) \cap I$. But, this implies $f^{-1}(U)$ and $f^{-1}(V)$ disconnect $I$, which is a contradiction. Therefore, $f(I)$ is connected. 

**Corollary 6.26** (Intermediate Value Theorem). If $f : [a, b] \to \mathbb{R}$ is continuous and $\alpha$ is between $f(a)$ and $f(b)$, then there is a $c \in [a, b]$ such that $f(c) = \alpha$.

**Proof.** This is an easy consequence of Theorem 6.25 and Theorem 5.14. 

**Definition 6.27.** A function $f : D \to \mathbb{R}$ has the Darboux property if whenever $a, b \in D$ and $\gamma$ is between $f(a)$ and $f(b)$, then there is a $c$ between $a$ and $b$ such that $f(c) = \gamma$.

Calculus texts usually call the Darboux property the intermediate value property. Corollary 6.26 shows that a function continuous on an interval has the Darboux property. The next example shows continuity is not necessary for the Darboux property to hold.

**Example 6.21.** The function 

$$f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is not continuous, but does have the Darboux property. (See Figure 4.) It can be seen from Example 6.6 that $0 \notin C(f)$.

To see $f$ has the Darboux property, choose two numbers $a < b$.

If $a > 0$ or $b < 0$, then $f$ is continuous on $[a, b]$ and Corollary 6.26 suffices to finish the proof.

On the other hand, if $0 \in [a, b]$, then there must exist an $n \in \mathbb{Z}$ such that both $$\frac{2}{(4n+1)\pi}, \frac{2}{(4n+3)\pi} \in [a, b].$$ Since $f\left(\frac{2}{(4n+1)\pi}\right) = 1$, $f\left(\frac{2}{(4n+3)\pi}\right) = -1$ and $f$ is continuous on the interval between them, we see $f([a, b]) = [-1, 1]$, which is the entire range of $f$. The claim now follows.

6. Uniform Continuity

Most of the ideas contained in this section will not be needed until we begin developing the properties of the integral in Chapter 8.

**Definition 6.28.** A function $f : D \to \mathbb{R}$ is uniformly continuous if for all $\varepsilon > 0$ there is a $\delta > 0$ such that when $x, y \in D$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

The idea here is that in the ordinary definition of continuity, the $\delta$ in the definition depends on both the $\varepsilon$ and the $x$ at which continuity is being tested. With uniform continuity, $\delta$ only depends on $\varepsilon$; i.e., the same $\delta$ works uniformly across the whole domain.

**Theorem 6.29.** If $f : D \to \mathbb{R}$ is uniformly continuous, then it is continuous.
Section 6: Uniform Continuity

**Theorem 6.30.** If \( f : D \to \mathbb{R} \) is continuous and \( D \) is compact, then \( f \) is uniformly continuous.

**Proof.** Suppose \( f \) is not uniformly continuous. Then there is an \( \varepsilon > 0 \) such that for every \( n \in \mathbb{N} \) there are \( x_n, y_n \in D \) with \( |x_n - y_n| < 1/n \) and \( |f(x_n) - f(y_n)| \geq \varepsilon \). An application of the Bolzano-Weierstrass theorem yields a subsequence \( x_{n_k} \) of \( x_n \) such that \( x_{n_k} \to x_0 \in D \).

Since \( f \) is continuous at \( x_0 \), there is a \( \delta > 0 \) such that whenever \( x \in (x_0 - \delta, x_0 + \delta) \cap D \), then \( |f(x) - f(x_0)| < \varepsilon/2 \). Choose \( n_k \in \mathbb{N} \) such that \( 1/n_k < \delta/2 \) and \( x_{n_k} \in (x_0 - \delta/2, x_0 + \delta/2) \). Then both \( x_{n_k}, y_{n_k} \in (x_0 - \delta, x_0 + \delta) \) and

\[
\varepsilon \leq |f(x_{n_k}) - f(y_{n_k})| = |f(x_{n_k}) - f(x_0) + f(x_0) - f(y_{n_k})| \\
\leq |f(x_{n_k}) - f(x_0)| + |f(x_0) - f(y_{n_k})| < \varepsilon/2 + \varepsilon/2 = \varepsilon,
\]

which is a contradiction.

Therefore, \( f \) must be uniformly continuous.

The following corollary is an immediate consequence of Theorem 6.30.

**Corollary 6.31.** If \( f : [a, b] \to \mathbb{R} \) is continuous, then \( f \) is uniformly continuous.

**Theorem 6.32.** Let \( D \subset \mathbb{R} \) and \( f : D \to \mathbb{R} \). If \( f \) is uniformly continuous and \( x_n \) is a Cauchy sequence from \( D \), then \( f(x_n) \) is a Cauchy sequence.

**Proof.** The proof is left as Exercise 6.37.

Uniform continuity is necessary in Theorem 6.32. To see this, let \( f : (0,1) \to \mathbb{R} \) be \( f(x) = 1/x \) and \( x_n = 1/n \). Then \( x_n \) is a Cauchy sequence, but \( f(x_n) = n \) is not. This idea is explored in Exercise 6.32.

It's instructive to think about the converse to Theorem 6.32. Let \( f(x) = x^2 \), defined on all of \( \mathbb{R} \). Since \( f \) is continuous everywhere, Corollary 6.12 shows \( f \) maps Cauchy sequences to Cauchy sequences. On the other hand, in Exercise 6.36, it is shown that \( f \) is not uniformly continuous. Therefore, the converse to Theorem 6.32 is false. Those functions mapping Cauchy sequences to Cauchy sequences are sometimes said to be *Cauchy continuous*. The converse to Theorem 6.32 can be tweaked to get a true statement.

**Theorem 6.33.** Let \( f : D \to \mathbb{R} \) where \( D \) is bounded. If \( f \) is Cauchy continuous, then \( f \) is uniformly continuous.
Suppose $f$ is not uniformly continuous. Then there is an $\varepsilon > 0$ and sequences $x_n$ and $y_n$ from $D$ such that $|x_n - y_n| < 1/n$ and $|f(x_n) - f(y_n)| \geq \varepsilon$. Since $D$ is bounded, the sequence $x_n$ is bounded and the Bolzano-Weierstrass theorem gives a Cauchy subsequence, $x_{n_k}$. The new sequence $z_k = \begin{cases} x_{n(k+1)/2} & k \text{ odd} \\ y_{n_k/2} & k \text{ even} \end{cases}$ is easily shown to be a Cauchy sequence. But, $f(z_k)$ is not a Cauchy sequence, since $|f(z_k) - f(z_{k+1})| \geq \varepsilon$ for all odd $k$. This contradicts the fact that $f$ is Cauchy continuous. We’re forced to conclude the assumption that $f$ is not uniformly continuous is false.

7. Exercises

6.1. Prove $\lim_{x \to -2} (x^2 + 3x) = -2$.

6.2. Give examples of functions $f$ and $g$ such that neither function has a limit at $a$, but $f + g$ does. Do the same for $fg$.

6.3. Let $f : D \to \mathbb{R}$ and $a \in D'$.

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a} f(x) = \lim_{x \to a} f(x) = L$$

6.4. Find two functions defined on $\mathbb{R}$ such that

$$0 = \lim_{x \to 0} (f(x) + g(x)) \neq \lim_{x \to 0} f(x) + \lim_{x \to 0} g(x).$$

6.5. If $\lim_{x \to a} f(x) = L > 0$, then there is a $\delta > 0$ such that $f(x) > 0$ when $0 < |x - a| < \delta$.

6.6. If $Q = \{q_n : n \in \mathbb{N}\}$ is an enumeration of the rational numbers and

$$f(x) = \begin{cases} 1/n, & x = q_n \\ 0, & x \in \mathbb{Q}^c \end{cases}$$

then $\lim_{x \to a} f(x) = 0$, for all $a \in \mathbb{Q}^c$.

6.7. Use the definition of continuity to show $f(x) = x^2$ is continuous everywhere on $\mathbb{R}$.

6.8. Prove that $f(x) = \sqrt{x}$ is continuous on $[0, \infty)$.

6.9. If $f$ is defined as in (50), then $D = C(f)^c$.

6.10. If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then there is a countable set $D$ such that the values of $f$ can be altered on $D$ in such a way that the altered function is left-continuous at every point of $\mathbb{R}$. 

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7. EXERCISES

6.11. Does there exist an increasing function \(f : \mathbb{R} \rightarrow \mathbb{R}\) such that \(C(f) = Q\)?

6.12. If \(f : \mathbb{R} \rightarrow \mathbb{R}\) and there is an \(a > 0\) such that \(|f(x) - f(y)| \leq a|x - y|\) for all \(x, y \in \mathbb{R}\), then show that \(f\) is continuous.

6.13. Suppose \(f\) and \(g\) are each defined on an open interval \(I\), \(a \in I\) and \(a \in C(f) \cap C(g)\). If \(f(a) > g(a)\), then there is an open interval \(J\) such that \(f(x) > g(x)\) for all \(x \in J\).

6.14. If \(f, g : (a, b) \rightarrow \mathbb{R}\) are continuous, then \(G = \{x : f(x) < g(x)\}\) is open.

6.15. If \(f : \mathbb{R} \rightarrow \mathbb{R}\) and \(a \in C(f)\) with \(f(a) > 0\), then there is a neighborhood \(G\) of \(a\) such that \(f(G) \subset (0, \infty)\).

6.16. Let \(f\) and \(g\) be two functions which are continuous on a set \(D \subset \mathbb{R}\). Prove or give a counter example: \(\{x \in D : f(x) > g(x)\}\) is relatively open in \(D\).

6.17. If \(f, g : \mathbb{R} \rightarrow \mathbb{R}\) are functions such that \(f(x) = g(x)\) for all \(x \in Q\) and \(C(f) = C(g) = \mathbb{R}\), then \(f = g\).

6.18. Let \(I = [a, b]\). If \(f : I \rightarrow I\) is continuous, then there is a \(c \in I\) such that \(f(c) = c\).

6.19. Find an example to show the conclusion of Problem 6.18 fails if \(I = (a, b)\).

6.20. If \(f\) and \(g\) are both continuous on \([a, b]\), then \(\{x : f(x) \leq g(x)\}\) is compact.

6.21. If \(f : [a, b] \rightarrow \mathbb{R}\) is continuous, not constant,

\[
m = \text{glb}\{f(x) : a \leq x \leq b\} \quad \text{and} \quad M = \text{lub}\{f(x) : a \leq x \leq b\},
\]

then \(f([a, b]) = [m, M]\).

6.22. Suppose \(f : \mathbb{R} \rightarrow \mathbb{R}\) is a function such that every interval has points at which \(f\) is negative and points at which \(f\) is positive. Prove that every interval has points where \(f\) is not continuous.

6.23. If \(f : [a, b] \rightarrow \mathbb{R}\) has a limit at every point, then \(f\) is bounded. Is this true for \(f : (a, b) \rightarrow \mathbb{R}\)?

6.24. Give an example of a bounded function \(f : \mathbb{R} \rightarrow \mathbb{R}\) with a limit at no point.

6.25. If \(f : \mathbb{R} \rightarrow \mathbb{R}\) is continuous and periodic, then there are \(x_m, x_M \in \mathbb{R}\) such that \(f(x_m) \leq f(x) \leq f(x_M)\) for all \(x \in \mathbb{R}\). (A function \(f\) is periodic, if there is a \(p > 0\) such that \(f(x + p) = f(x)\) for all \(x \in \mathbb{R}\). The least such \(p\) is called the period of \(f\).)

6.26. A set \(S \subset \mathbb{R}\) is disconnected iff there is a continuous \(f : S \rightarrow \mathbb{R}\) such that \(f(S) = [0, 1]\).
6.27. If \( f : \mathbb{R} \to \mathbb{R} \) satisfies \( f(x + y) = f(x) + f(y) \) for all \( x \) and \( y \) and \( 0 \in C(f) \), then \( f \) is continuous.

6.28. Assume that \( f : \mathbb{R} \to \mathbb{R} \) is such that \( f(x + y) = f(x)f(y) \) for all \( x, y \in \mathbb{R} \). If \( f \) has a limit at zero, prove that either \( \lim_{x \to 0} f(x) = 1 \) or \( f(x) = 0 \) for all \( x \in \mathbb{R} \setminus \{0\} \).

6.29. If \( F \subseteq \mathbb{R} \) is closed, then there is an \( f : \mathbb{R} \to \mathbb{R} \) such that \( F = C(f) \).

6.30. If \( f : [a, b] \to \mathbb{R} \) is uniformly continuous, then \( f \) is continuous.

6.31. A function \( f : \mathbb{R} \to \mathbb{R} \) is \emph{periodic} with period \( p > 0 \), if \( f(x + p) = f(x) \) for all \( x \).
If \( f : \mathbb{R} \to \mathbb{R} \) is periodic with period \( p \) and continuous on \([0, p] \), then \( f \) is uniformly continuous.

6.32. Prove that an unbounded function on a bounded open interval cannot be uniformly continuous.

6.33. If \( f : D \to \mathbb{R} \) is uniformly continuous on a bounded set \( D \), then \( f \) is bounded.

6.34. Prove Theorem 6.29.

6.35. Every polynomial of odd degree has a root.

6.36. Show \( f(x) = x^2 \), with domain \( \mathbb{R} \), is not uniformly continuous.

6.37. Prove Theorem 6.32.